Geometric phase for Jaynes-Cummings model and interference of atomic states

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An adiabatic cyclic evolution of the parameters in the model of an interaction of a two-level system with a resonator mode leads to a nontrivial phase shift of a wave function [Berry phase; Proc. R. Soc. A **392**, 45 (1984)]. This phase, geometric in nature, could in principle be observed experimentally.

The exactly solvable Jaynes-Cummings model[2] describes the interaction of a resonator mode with a two-level system in the rotating-wave approximation. This model is used to describe so-called Rydberg masers,[3,4] in which masing occurs on highly excited electronic states of individual atoms in a high-Q resonator. The Hamiltonian of the

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model of Ref. 2 can be written in the form

$$H = \omega_0 a^+ a + \omega_1 \sigma_3 + \mu a^+ \sigma_- + \mu^* a \sigma_+ , \tag{1}$$

where a^+ and a are operators which respectively create and annihilate a photon in the resonator mode, $\sigma_\pm$ and σ_3 are the Pauli matrices, ω_0 is the frequency of the mode, ω_1 is the transition frequency in the two-level system, and μ is the coupling constant. The magnitude of the coupling constant, $|\mu| = \rho$, has the meaning of the Rabi frequency of one photon, and the argument, φ, has the meaning of the initial phase shift between the atomic polarization and the electromagnetic field.[5]

The eigenfunctions of the Jaynes-Cummings model are

$$\Psi = \alpha | N, 0\rangle + \beta | N - 1,1\rangle , \tag{2}$$

where $|N,0\rangle$ is a state with N photons in the mode and an unexcited two-level system, while $|N-1, 1\rangle$ is a state with $N-1$ photons in the mode and an excited two-level system.

Under the assumption that the parameters of the model vary adiabatically (in practice, this situation can be arranged by mechanically changing the volume and geometry of the resonator[6] and by means of a Stark shift of the transition frequency), i.e., $\Delta = \omega_1 - \omega_0 \equiv \Delta(\tau)$, we have $\mu, \mu^* = \mu(\tau), \mu^*(\tau)$, so that $\dot{\Delta}/\Delta$ and $(\dot{\mu}/\mu)$ are the slowest characteristic frequencies in the system.

For the quantities $u(\tau)$, $v(\tau)$, defined in terms of $\alpha(\tau)$, $\beta(\tau)$ by

$$\begin{aligned} u &= \exp\left\{ -\frac{1}{2i} \int^t \left[(2N-1)\omega_0 + i\frac{\dot{\mu}}{\mu}\right] d\tau \right\} \alpha , \\ v &= \exp\left\{ -\frac{1}{2i} \int^t \left[(2N-1)\omega_0 + i\frac{\dot{\mu}^*}{\mu^*}\right] d\tau \right\} \beta , \end{aligned} \tag{3}$$

we then find the following expressions in the WKB approximation:

$$u = \frac{a_1}{\sqrt{\Omega}} e^{\int^t (-f(\tau) + i\varphi(\tau)) d\tau} + \frac{a_2}{\sqrt{\Omega}} e^{\int^t (f(\tau) - i\varphi(\tau)) d\tau}$$

$$v = \frac{b_1}{\sqrt{\Omega}} e^{\int^t (f + i\varphi) d\tau} + \frac{b_2}{\sqrt{\Omega}} e^{-\int^t (f + i\varphi) d\tau},$$

$$f(\tau) = \frac{1}{4\Omega}\left(\dot{\Delta} - \Delta\frac{\dot{\rho}}{\rho}\right), \quad \varphi(\tau) = \Omega + \frac{\Delta\dot{\varphi}}{4\Omega}, \quad \Omega = (4|\mu|^2 N + \Delta^2)^{1/2}. \tag{4}$$

Here N is the total number of excitations in the system (an integral of motion), and a_1, a_2, b_1, and b_2 are constants which depend on the initial conditions.

For the initial conditions corresponding to an unexcited two-level system we have

$$u(\tau) = \sqrt{\frac{\Omega(0)}{\Omega(\tau)}}\ [\cosh\gamma\cos\delta - i\sinh\gamma\sin\delta]$$

$$\upsilon(\tau) = \sqrt{\frac{\Omega(0)}{\Omega(\tau)}}\ [\sinh\gamma\cos\delta + i\cosh\gamma\sin\delta], \qquad (5)$$

where

$$\gamma = \int^{t} \frac{1}{4\Omega}(\dot{\Delta} - \Delta\frac{\dot{\rho}}{\rho})d\tau, \quad \delta = \delta_1 + \delta_2, \quad \delta_1 = \int^{t} \Omega d\tau, \quad \delta_2 = \int^{t} \frac{\Delta\dot{\varphi}}{4\Omega} d\tau .$$

The angles γ and δ_1 are dynamic in nature (see Ref. 7 regarding the difference between dynamic and topological phases). If the evolution of the system is cyclic, $\Delta(T) = \Delta(0)$, $\mu(T) = \mu(0)$, $\int_0^T \Omega(\tau)d\tau = 2\pi m$, where m in an integer, then these angles are identically zero mod(2π). In contrast, the angle δ_2 may be nonzero upon a cyclic change in the parameters. The wave function of the system in this case is

$$|\Psi\rangle = a|N,0\rangle + ib|N-1,1\rangle , \qquad (6)$$

where $a = \cos\theta$, $b = \sin\theta$, and $\theta \equiv \delta_2$.

In this model the angle θ has the meaning of a topological phase. It has two remarkable (but quite natural) properties. In the limiting case $N = 0$, we have $\theta = 0$; i.e., the vacuum state does not acquire a topological phase. In the other limit we have $\theta \rightarrow (1/\sqrt{N}) \sim 0$; i.e., the effect is of a purely quantum nature and disappears in the limit of a classical field ($N = \infty$). In the case of an exact resonance between the atomic transition and the resonator frequency, it is also zero.

The numerical value of Berry's phase is not small. For a sinusoidal variation of the frequency deviation, the phase, and the amplitude of the coupling constant, with $\Delta = \pm\epsilon\cos(\gamma_R t), \varphi = \sin(\gamma_R t), 2\sqrt{N}\rho = \epsilon\sin(\gamma_R t)\gamma_R \ll \Delta,\rho$, for example, we would have $\theta = \pm\pi/4$.

In principle, Berry's phase could be measured in Ramsey's spectroscopic arrangement[8] with two separated fields. The application of that arrangement to Rydberg systems was described in Ref. 9. Specifically, in the course of a measurement a wave function of the type $|\Psi\rangle = \alpha_0|a\rangle + \beta_0|b\rangle$ is transformed into a wave function $|\psi'\rangle = \tilde{\alpha}_0|a\rangle + \tilde{\beta}_0 e^{i\phi}|b\rangle$, where $\tilde{\alpha}_0$ and $\tilde{\beta}_0$ are the altered amplitudes, and ϕ is the phase of a (reference) field. Ramsey's arrangement makes it possible to measure the quantity $|\tilde{\alpha}_0\tilde{\beta}_0^*|\cos(\theta + \phi - \omega_1 t_0)$, where $\theta = \arg(\alpha_0\beta_0^*) = \arg(\tilde{\alpha}_0\tilde{\beta}_0^*)$, and t_0 is the transit time between the two separated fields. In Ramsey's arrangement, the roles of the interferometer arms are played by the excited and unexcited atomic states. The preparation of an N-photon (Fok) state of the electromagnetic field in the resonator (quantum fluctuations in the number of photons mask the effect) would also be possible and was described in Ref. 10.

The Jaynes-Cummings model, a completely realistic model of supersymmetric quantum mechanics,[11] has a nontrivial behavior in the course of an adiabatic cyclic evolution of the parameters of the system. The geometric phase which arises during the cyclic evolution reaches a significant value ($|\theta| = (\pi/4) \approx 0.79$ in the example which we have been discussing here) and could in principle be measured in the atomic interferometer proposed by Krause *et al.*[9]

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