Finite-size effects in conformal theories and the nonlocal operators in one-dimensional quantum systems

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Finite-size effects have been used to determine the long-wavelength asymptotic behavior of the vacuum expectation values of certain nonlocal operators in one-dimensional quantum systems.

1. The new methods of the two-dimensional conformal field theory, which were developed to deal with the problems of string theory, have a variety of applications. Their equivalence to the theory of free fields, for example, has been verified for a large class of conformal theories (developed by the so-called GKO construction), which has greatly simplified the calculation of the correlation functions of local conformal operators. On the other hand, the string compactification is closely connected with the determination of the long-wavelength asymptotic behavior of various vacuum expectation values in one-dimensional quantum models. A method of determining the long-wavelength asymptotic behavior of the correlation functions of various local operators in 1D quantum systems, based on the calculation of the finite-size effects in conformal theories, has recently been proposed.[1–7] The idea underlying this method is that at vanishing temperature such systems undergo a phase transition, and in the long-wavelength limit the system has conformal symmetry.[8]

Put more simply, the anomalous dimensions Δ and $\overline{\Delta}$ of the conformal operators $\phi_{\Delta,\overline{\Delta}}$ are related to the energy E_L^ϕ of the lowest excited states $|\phi\rangle$ which are such that $\langle \text{vac}|\phi|\phi\rangle \neq 0$

$$E_L^\phi - E_L^{\text{vac}} = 2\pi v h_\phi / L. \tag{1}$$

Here $h_\phi \equiv \Delta_\phi + \overline{\Delta}_\phi$ is the scaling dimension of the operator ϕ, L is the length of the system, v is the group velocity at the Fermi surface, and E_L^{vac} is the ground-state energy of the system. The long-wavelength asymptotic behavior of the correlation function of the fields $\phi_{\Delta,\overline{\Delta}}$ has the form

$$\langle \phi(x)|\phi(0)\rangle \sim \cos(P^{\phi}x)x^{-2h_\phi}, \tag{2}$$

where P^{ϕ} is the momentum of the state $|\phi\rangle$, which is nonvanishing in the presence of a gap in the momentum-operator spectrum.

Similarly, the central charge c of the Virasoro algebra which arises is related to the volume correction, $\sim L^{-1}$, to the ground-state energy of the system.[3,4] For most one-component systems (such as a one-dimensional Bose gas or Fermi gas, the Heisenberg magnet with a spin of 1/2, etc.) $c=1$ (Refs. 2, 5, and 6). The theories mentioned above thus fall into the general class of the one-component Gaussian model.[9] The spectrum of the anomalous dimensions of the local (primary) operators $\phi_{n,m}$ in the Gaussian model depends on one continuous parameter R and has the form

$$h_\phi = n^2/R^2 + m^2R^2/4, \ n, m - \text{integers}. \tag{3}$$

The parameter R depends on the parameters of the original model: $R^2 = 8\pi N/vL$, where N is the number of particles in the system (or the number of flipped spins in the case of a magnet). In the thermodynamic limit we have $N\to\infty$, $L\to\infty$, and $N/L=\rho=\text{const}$.

2. We will show that the correlation functions of some nonlocal operators in 1D systems can be found by using similar methods. Let us first consider a XXZ Heisenberg antiferromagnet described by the Hamiltonian

$$\hat{H}_1 = -1/2 \sum_{i=1}^{L} (\sigma_i^1\sigma_{i+1}^1 + \sigma_i^2\sigma_{i+1}^2 - \cos\gamma\cdot\sigma_i^3\sigma_{i+1}^3), \ 0\leqslant\gamma<\pi. \tag{4}$$

It would be of interest[10,11] to find the antiferromagnetic-vacuum expectation values of the nonlocal operators $S_{xy}\equiv\Pi_{j=x}^{y}\sigma_j^3=\exp\{i\pi q(x,y)\}$, where $q(x,y)$ is an operator which gives the number of flipped spins at the sites from x to y and $T_{xy}=P_{xx+1}P_{x+1x+2}\dots P_{y-1y}P_{yx}$, where $P_{xy}\equiv(1+\vec{\sigma}_x\vec{\sigma}_y)/2$ is an operator for the interchange of spins at the x and y sites. The operator $T_{x,y}$ is an operator for a cycling interchange at the sites from x to y: $x\to x+1, x+1\to x+2,\dots,y-1\to y, y\to x$. Because of translational invariance, $\langle\text{vac}|S_{xy}|\text{vac}\rangle$ and $\langle\text{vac}|T_{xy}|\text{vac}\rangle$ depend exclusively on $x-y$.

We will consider the Hilbert space which simultaneously incorporates all the states of the spin chain with different number of sites and with various boundary conditions. We introduce the operators which vary the number of sites by unity: the operator $a_{x\alpha}^{+}$ ($\alpha=\pm1$) creates a site with spin α between the sites of the original chain with indices x and $x+1$ and the operator $b_{x\alpha}$ which annihilates site x (with spin β_x) in the case $\beta_x=\alpha$ and which causes the states in which $\beta_x=-\alpha$ to vanish. The operators $a_{x\alpha}^{+}$ thus operate from a sector with L number of sites into a sector with $L+1$ number of sites. In contrast, the operators $b_{x\alpha}$ decrease the number of sites by unity. We also introduce the operator $s_x\equiv\Pi_{j=x}\sigma_j^3$ which links the sectors with the periodic and antiperiodic boundary conditions. Clearly,

$$T_{xy}=\sum_\alpha a_{x\alpha}^{+}b_{y\alpha},\ S_{xy}=S_xS_y. \tag{5}$$

We use $|L,+\rangle$ ($|L,-\rangle$) to denote the vacuum of the chain comprised of L sites with periodic (antiperiodic) boundary conditions (in the thermodynamic limit these states are identical, but for now the corrections to L^{-1} are important). We will then have $\langle L-1,+|b_{x\alpha}|L,+\rangle \neq 0, \langle L+1,+|a^{+}_{x\alpha}|L,+\rangle \neq 0, \langle L,-|S_x|L,+\rangle \neq 0$. We can use Eq. (2), after finding the dimensions of the operators $b_{x\alpha}$ and S_x from (1), to determine the asymptotic behavior of the expectation values of the operators (5) for large $|x-y|$. We must therefore calculate the shift in the energies of the states $|L+1,+\rangle$ and $|L,-\rangle$, in contrast with $|L,+\rangle$, and we must also calculate their momenta. We see that the spectrum found in this manner is given by Eq. (3) with the half-integers n and m. In particular, $b_{\pm 1}$ can be identified with the nonlocal operators $\phi_{\pm 1/2,0}$ in an expanded Gaussian model, and S can be identified with $\phi_{0,1/2}$. As a result, we find

$$\langle \text{vac}|\exp\{i\pi q(x,y)\}|\text{vac}\rangle \sim \cos[\pi(x-y)/2]\cdot|x-y|^{-\lambda}, \tag{6a}$$

$$\langle \text{vac}|T_{xy}|\text{vac}\rangle \sim |x-y|^{-\mu}. \tag{6b}$$

Here $\lambda = R^2/8$, and $\mu = R^{-2}/2$, where $R^2 = 2\pi(\pi-\gamma)^{-1}$ coincides with the inverse scaling dimension of the operators σ^1 and σ^2 (Refs. 2 and 5).

In the isotropic case of the XXX model [$\gamma=0$ in (4)] the operator S_x has a scaling dimension 1/8 and is a spin field.[12]

3. The same method operates in a one-dimensional Bose model or Fermi gas model which is described by a continuous Hamiltonian

$$\hat{H}_2 = \int_0^L dx\, \partial_x \psi^*(x)\, \partial_x \psi(x) + \frac{1}{2}\cdot g \int\int_0^L dx dy\cdot \psi^*(x)\psi^*(y)V(x-y)\psi(x)\psi(y), \tag{7}$$

where $V(x)$ is a pairwise-interaction (repulsion) potential of a fairly general type. The analog of the operator S_{xy} is given by the same formula $S_{xy} = \exp[i\pi q(x,y)]$, where $q(x,y)$ is now a particle number operator in the interval from x to y. At $|x-y| \gg L/N = \rho^{-1}$ we find

$$\langle \text{vac}|S_{xy}|\text{vac}\rangle \sim \cos(\pi\rho|x-y|)\cdot|x-y|^{-1/4\theta}, \tag{8}$$

where $\theta = 2R^2 = v(4\pi\rho)^{-1}$ is the critical index of the correlation function of the boson fields ψ (Ref. 10), ρ is the density, and v is the group velocity at the Fermi surface.

The operator S takes the Jordan–Wigner transform from the boson operators ψ_B to the fermion operators ψ_F: $\psi F(x) = \psi_F^*(x) = \psi_B(x)S_{xL}$ and $\psi_F^*(x) = S^*_{xL}\psi_B^*(x)$. We can thus idenify ψ_F with the operator $\phi_{1,1/2}$ in the Gaussian model and we can calculate the asymptotic behavior of the correlation function of the fermion-model fields (7)

$$\langle \text{vac}|\psi_F^*(x)\psi_F(y)|\text{vac}\rangle \sim \cos(\pi\rho|x-y|)\cdot|x-y|^{-\theta-(1/4)\theta}. \tag{9}$$

In general, the dimension spectrum of the fermion model (7) is determined by Eq. (3) with the following condition imposed on n and m: when n is even, m is an integer and when n is odd, m is a half-integer.

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