Relativistic Coulomb problem and bound states in continuum

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A close relationship is traced between positive levels of the relativistic Coulomb problem and Wigner–Von Neumann states immersed in the continuum.

A recent study[1] of what appeared to be a well-known problem—the relativistic Coulomb interaction of two particles of identical mass—through a numerical solution of a quasipotential equation, revealed an entire system of bound states with positive binding energies ($\varepsilon > 0$). Another unexpected result was that new states appeared in the region $\varepsilon > 0$ in systems of charged particles of both like and unlike charge. The well-known Coulomb levels in a system of unlike charges were reproduced in the region of negative binding energies ($\varepsilon < 0$), as they should have been, while the system of like charges revealed no discrete levels.

These results made possible a consistent interpretation[1] of some experimental data—which seemed at first glance totally unrelated—on new e^+e^- resonances observed in heavy-ion collisions (GSI resonances; Ref. 2, for example) and some unusual diproton resonances which have been observed.[3,4] It is important to note that the numerical predictions of these levels were found through the use of only the fundamental constants $\alpha = 1/137$ and the electron and proton masses as parameters.

In this letter we would like to call attention to an analogy between the problem discussed in Ref. 1 and Wigner and Von Neumann's well-known example[5] of the existence of bound states immersed in the continuum.

A quasipotential equation of the type

$$2\omega(M-2\omega)\varphi(\mathbf{p}) = \frac{1}{(2\pi)^3} \int \frac{d\mathbf{p}'}{2\omega'} V(\mathbf{p},\mathbf{p}'|M)\varphi(\mathbf{p}') \quad (1)$$

was studied in Ref. 1, where $\omega = \sqrt{\mathbf{p}^2 + m^2}$, $\omega' = \sqrt{\mathbf{p}'^2 + m^2}$, M is the mass of the bound state, and the quasipotential is

$$V(\mathbf{p},\mathbf{p}'|M) = \frac{(2me)^2}{|\mathbf{q}|(M-\Omega+io)}, \quad (2)$$

where $\mathbf{q} = \mathbf{p}' - \mathbf{p}$, $\Omega = \omega + \omega' + |\mathbf{q}|$, and $e^2/4\pi \equiv \alpha$.

Equation (1) with quasipotential (2) describes the interaction of two unlike charged particles of mass m in the approximation of the exchange of a single massless neutral scalar photon. An equation describing the interaction of two charged particles of like charge can be found from (1) and (2) through the substitution $\alpha \to -\alpha$. A potential of the type in (2) had been found previously in Refs. 6 and 7 and also, for the

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case of spinor electrodynamics, in Ref. 8. Expression (2) leads to a nonlocal energy-dependent expression for the potential in coordinate space. However, if we have $|p|$, $|p'| \ll m$ in (2), potential (2) takes the simpler form

$$V(\mathbf{p}, \mathbf{p}' \,|\, M) = \frac{(2me)^2}{|\mathbf{q}|(\epsilon - |\mathbf{q}| + io)}, \tag{3}$$

where $\varepsilon = M - 2m$ is the binding energy. In the region of bound states ($\varepsilon < 0$) the denominator in (3) does not lead to any additional singularities, and the limit $\varepsilon \to 0$ yields the nonrelativistic Coulomb expression for the potential, $-(2me)^2/|\mathbf{q}|^2$. The behavior of potential (3) in coordinate space was studied for arbitrary ε in Ref. 7. At $\varepsilon > 0$, expression (3) has a singularity at $\varepsilon = |\mathbf{q}|$, and taking the nonrelativistic limit becomes problematical. The rule for circumventing the pole in (2) and (3) is very important and follows directly from Feynman's rules for circumventing poles in the propagators of interacting particles.[1] Taking this approach guarantees correct casual properties for potentials (2) and (3). In this approximation, Eq. (1) can be rewritten as a Schrödinger equation in momentum space:

$$(\epsilon - \frac{p^2}{m})\varphi(\mathbf{p}) = \frac{e^2}{(2\pi)^3} \int d\mathbf{p}' \frac{1}{|\mathbf{q}|(\epsilon - |\mathbf{q}| + io)} \varphi(\mathbf{p}'). \tag{4}$$

Transforming to coordinate space in (4), and taking into account the dependence of the quasipotential on only the momentum difference, we find the Schrödinger equation

$$[-\frac{\vec{\nabla}^2}{m} + V(r)]\tilde{\varphi}(\mathbf{r}) = \epsilon\tilde{\varphi}(\mathbf{r}) \tag{5}$$

with a potential

$$V(r) = -\frac{2\alpha}{\pi}\frac{1}{r}\{\cos(\epsilon r)[\mathrm{si}(\epsilon r) + \pi] - \sin(\epsilon r)\,\mathrm{ci}(\epsilon r) + i\pi\sin(\epsilon r)\}, \tag{6}$$

where $\mathrm{si}(x)$ and $\mathrm{ci}(x)$ are the integral sine and integral cosine, respectively. A similar potential was derived for the case of spinor electrodynamics in Ref. 8.

We would like to call attention to a key property of potential (6), specifically, the oscillatory behavior of its real part in the limit $r \to \infty$:

$$\mathrm{Re}\, V(r) \underset{r \to \infty}{\to} -2\alpha\frac{\cos(\epsilon r)}{r}. \tag{7}$$

It is the property—this oscillatory behavior at infinity against the background of a slow $1/r$ decay—which is the most characteristic feature of this potential and which is actually responsible for the possible appearance of levels with a positive binding energy ε.

From the mathematical standpoint, the situation which arises here is analogous to the classical Wigner–Von Neumann example.[5] As we know, Wigner and Von Neumann explicitly constructed a potential with an asymptotic behavior $-8\sin(2r)/r$, in which there exists an energy level with a positive binding energy, $\varepsilon = 1$ in Coulomb units; i.e., there exists a positive level immersed in the continuum.

Von Neumann and Wigner themselves[5] constructed only one level with a positive binding energy. Other examples of levels with $\varepsilon > 0$ were found subsequently for potentials of the Wigner–Von Neumann type. In particular, a level with $\varepsilon = 4$ in Coulomb units was reported in Ref. 9. That level lies above all positive maxima of the potential. As we have already mentioned, we have derived[1] an entire system of levels with $\varepsilon > 0$ for the more complicated case of potential (2).

The physical meaning of the appearance of positive levels in oscillating potentials is that at certain values of the energy coherent reflections occur from potential hills, with the result that the wave going off to infinity decays.[10]

The Wigner–Von Neumann example has been perceived as a purely mathematical result, devoid of any real physical meaning. In contrast, we have shown that an analysis of the bound-state problem in quantum field theory may reveal potentials in which there exist discrete levels with a positive binding energy, which are analogous to the levels immersed in the continuum in examples of the Wigner–Von Neumann type. We wish to stress once more that from the physical standpoint, the existence of such an unusual spectrum of states is related to the circumstance that the interaction of two relativistic charged particles is described by an energy-dependent oscillating potential.

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[1]B. A. Arbuzov *et al.*, Preprint 89–1/78, Scientific-Research Institute of Nuclear Physics, Moscow State University, Moscow; in: *Problems of High-Energy Physics and Field Theory* (ed. V. A. Petrov), Nauka, Moscow, 1989, p. 362.
[2]T. Cowan *et al.*, Phys. Rev. Lett. **56**, 444 (1986).
[3]Yu. A. Troyan *et al.*, Preprint D1-88-399, Joint Institute for Nuclear Research, 1988, Dubna.
[4]O. B. Abdinov *et al.*, Preprint P1-88-102, Joint Institute for Nuclear Research, 1988, Dubna.
[5]J. Von Neumann and E. Wigner, Phys. Z. **30**, 365 (1929).
[6]V. G. Kadyshevski, Nucl. Phys. B **6**, 125 (1968).
[7]V. N. Kapshaĭ *et al.*, Teor. Mat. Fiz. **69**, 400 (1986).
[8]A. A. Arkhipov, Teor. Mat. Fiz. **74**, 69 (1988).
[9]F. H. Stillinger and D. R. Herrick, Phys. Rev. A **11**, 446 (1975).
[10]M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 4, Academic, Orlando, 1980.

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