New algebra of internal symmetries of Maxwell's equations

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A new algebra of the invariance of the homogeneous Maxwell's equations is constructed. It contains a 16-dimensional Lie algebra, a Grassmann algebra, and a superalgebra as subalgebras.

Internal symmetries contain important information about entities of research interest. As an example one might cite the dual symmetry of the equations of electrodynamics, which implies the existence of not only electric but also magnetic charges.[1] The internal symmetries of the equations of the free electromagnetic field have been studied by several investigators.[1–5] So far, the research has been restricted to the class of Lie algebras and groups. As an example one might cite the Lie algebra of the $U(2) \otimes U(2)$ group.[2] Our purpose in the present letter is to extend the class of algebras of the invariance of Maxwell's equations, in particular, to construct a Grassman algebra and a superalgebra.

We work from a vector formulation of the equations:

$$\nabla \times \mathbf{E} = -\partial_0 \mathbf{H}; \quad \nabla \cdot \mathbf{E} = 0; \quad \nabla \times \mathbf{H} = \partial_0 \mathbf{E}; \quad \nabla \cdot \mathbf{H} = 0, \tag{1}$$

where $x^0 = ct$ (c is the velocity of light, and t is the time), and x, y, z are spatial variables. Using Fourier transforms of the fields $\mathbf{E}$ and $\mathbf{H}$,

$$\mathbf{E} = (1/2\pi)^{3/2} \int d^3p \tilde{\mathbf{E}}(\mathbf{p}, t) e^{i\mathbf{p}\cdot\mathbf{x}}; \qquad \mathbf{H} = (1/2\pi)^{3/2} \int d^3p \tilde{\mathbf{H}}(\mathbf{p}, t) e^{i\mathbf{p}\cdot\mathbf{x}} \tag{2}$$

we can write Maxwell's equations in the momentum representation:

$$\mathbf{p} \times \tilde{\mathbf{E}} = i\partial_0 \tilde{\mathbf{H}}; \quad \mathbf{p} \cdot \tilde{\mathbf{E}} = 0; \quad \mathbf{p} \times \tilde{\mathbf{H}} = -i\partial_0 \tilde{\mathbf{E}}; \quad \mathbf{p} \cdot \tilde{\mathbf{H}} = 0 \cdot \tag{3}$$

We introduce a set of field transformations:

$$\begin{aligned}
\tilde{E}'_k &= \tilde{E}_k \cos\theta + i[(\not{D}^a)^L (\not{D}^s)^M (D^s)^N]_{kl} \tilde{E}_l \sin\theta; \\
\tilde{E}'_k &= \tilde{E}_k \cos\varphi + i[(\not{D}^a)^L (\not{D}^s)^M (D^s)^N]_{kl} \tilde{H}_l \sin\varphi; \\
\tilde{H}'_k &= \tilde{H}_k \cos\theta + i[(\not{D}^a)^L (\not{D}^s)^M (-D^s)^N]_{kl} \tilde{H}_l \sin\theta; \\
\tilde{H}'_k &= \tilde{H}_k \cos\varphi - i[(\not{D}^a)^L (\not{D}^s)^M (-D^s)^N]_{kl} \tilde{E}_l \sin\varphi,
\end{aligned} \tag{4}$$

where θ, φ are group parameters; $k, l = 1, 2, 3$; $L, M, N = 0, 1, 2, \ldots, n \to \infty$; and a and s specify antisymmetry and symmetry. We require that set (4) send Eqs. (3) into themselves. The corresponding invariance conditions make it possible to find the general form of the matrices $\mathcal{D}^a$, $\mathcal{D}^s$, D^s. They depend on the variable p_k and on certain functions of these variables.[3] They have the commutation and anticommutations properties

$$[\mathcal{D}^a, \mathcal{D}^s] = 0, \quad \{\mathcal{D}^a, D^s\} = 0, \quad [\mathcal{D}^s, D^s] = 0\ , \tag{5}$$

and they generate an infinite set of symmetry transformations (4) of Eqs. (3). We single out a subset of matrices which satisfy the normalization condition on the solutions

$$\mathcal{D}^a\mathcal{D}^a\varphi = E\varphi, \quad \mathcal{D}^s\mathcal{D}^s\varphi = E\varphi, \quad D^sD^s\varphi = E\varphi\ . \tag{6}$$

The normalization makes it possible to put the matrices in concrete form:

$$\mathcal{D}^a = \pm i \left\| \begin{matrix} 0 & -p_3/p & p_2/p \\ p_3/p & 0 & -p_1/p \\ -p_2/p & p_1/p & 0 \end{matrix} \right\| \tag{7}$$

$$\mathcal{D}^s = \pm \left\| \begin{matrix} (p^2-2p_1^2)/p^2 & -2p_1p_2/p^2 & -2p_1p_3/p^2 \\ -2p_1p_2/p^2 & (p^2-2p_2^2)/p^2 & -2p_2p_3/p^2 \\ -2p_1p_3/p^2 & -2p_2p_3/p^2 & (p^2-2p_3^2)/p^2 \end{matrix} \right\| \tag{8}$$

$$D^s = \pm \frac{2}{|L|} \left\| \begin{matrix} p_1^2p_2^2+p_1^2p_3^2-p_2^2p_3^2 & p_1p_2p_3^2 & p_1p_3p_2^2 \\ p_1p_2p_3^2 & p_1^2p_2^2-p_1^2p_3^2+p_2^2p_3^2 & p_2p_3p_1^2 \\ p_1p_3p_2^2 & p_2p_3p_1^2 & -p_1^2p_2^2+p_1^2p_3^2+p_2^2p_3^2 \end{matrix} \right\|, \tag{9}$$

where $p = (p_1^2+p_2^2+p_3^2)^{1/2}$, $|L| = 2(p_1^4p_2^4+p_1^4p_3^4+p_2^4p_3^4 - p^2p_1^2p_2^2p_3^2)^{1/2}$. Of these matrices, (7) and (9) with the plus sign were established in Ref. 2; this is the first realization of (8). With (7)–(9) in mind, we return to transformations (4). Corresponding to them is a set of infinitesimal 6×6 matrices

$$Y_{LMN} = i \left\| \begin{matrix} (\mathcal{D}^a)^L(\mathcal{D}^s)^M(D^s)^N & 0 \\ 0 & (\mathcal{D}^a)^L(\mathcal{D}^s)^M(-D^s)^N \end{matrix} \right\|,$$

$$Z_{LMN} = i \left\| \begin{matrix} 0 & (\mathcal{D}^a)^L(\mathcal{D}^s)^M(D^s)^N \\ -(\mathcal{D}^a)^L(\mathcal{D}^s)^M(-D^s)^N & 0 \end{matrix} \right\|. \tag{10}$$

These matrices have the permutation properties

$$[Y_{LMN}, Y_{L'M'N'}]_{\mp} = i((-1)^{NL'} \mp (-1)^{N'L}) Y_{L+L'M+M'N+N'} ;$$

$$[Y_{LMN}, Z_{L'M'N'}]_{\mp} = i((-1)^{NL'} \mp (-1)^{N'L+N}) Z_{L+L'M+M'N+N'} ; \qquad (11)$$

$$[Z_{LMN}, Z_{L'M'N'}]_{\mp} = i(-(-1)^{NL'+N'} \pm (-1)^{N'L+N}) Y_{L+L'M+M'N+N'} ,$$

and thus satisfy both a Lie algebra and a Grassmann algebra. They can be combined into a unified algebra $A^L_\infty \oplus A^G_\infty$ and a superalgebra $[Y,Y] = Y$, $[Y,Z] = Z$, $\{Z,Z\} = Y$. Using the multiplication table[3]

$$\mathcal{D}^a\mathcal{D}^a = \mathcal{D}^s, \quad \mathcal{D}^s\mathcal{D}^s = \mathcal{D}^s, \; D^sD^s = \mathcal{D}^s, \; \mathcal{D}^a\mathcal{D}^s = \mathcal{D}^a, \; \mathcal{D}^aD^s = D^s, \; \mathcal{D}^sD^s = D^s \qquad (12)$$

we single out 16 infinitesimal matrices Y_{000}, Y_{100}, Y_{010}, Y_{001}, Y_{110}, Y_{101}, Y_{011}, Y_{111}, Z_{000}, Z_{100}, Z_{010}, Z_{001}, Z_{110}, Z_{101}, Z_{011}, Z_{111}. In terms of these matrices we can express all of the other matrices in (10). The singled out matrices form closed sets and generate 16-dimensional Lie and Grassmann algebras, the direct sum $A^L_{16} \oplus A^G_{16}$, and the 16-dimensional superalgebra

$$[Y_{LMN}, Y_{L'M'N'}] = i((-1)^{NL'} - (-1)^{N'L}) Y_{L+L'M+M'N+N'} ;$$

$$[Y_{LMN}, Z_{L'M'N'}] = i((-1)^{NL'} - (-1)^{N'L+N}) Z_{L+L'M+M'N+N'} ; \qquad (13)$$

$$\{Z_{LMN}, Z_{L'M'N'}\} = i(-(-1)^{NL'+N'} - (-1)^{N'L+N}) Y_{L+L'M+M'N+N'} ;$$

$$L, \quad M, N = 0, 1.$$

The linear combinations

$$A_1 = -(iY_{101} + Z_{001})/4;$$

$$A_2 = -(Y_{100} + iZ_{000})/4; \quad A_3 = -(-Y_{001} + iZ_{101})/4;$$

$$A_4 = -(iY_{101} - Z_{001})/4;$$

$$A_5 = -(-Y_{100} + iZ_{000})/4; \quad A_6 = -(Y_{001} + iZ_{101})/4;$$

$$B_1 = -(iY_{111} + Z_{011})/4; \qquad (14)$$

$$B_2 = -(Y_{110} + iZ_{010})/4; \qquad B_3 = -(-Y_{011} + iZ_{111})/4;$$

$$B_4 = -(iY_{111} - Z_{011})/4;$$

$$B_5 = -(-Y_{110} + iZ_{010})/4; \quad B_6 = -(Y_{011} + iZ_{111})/4 ,$$

along with the commuting matrices Y_{000}, Y_{010}, Z_{100}, Z_{110}, form a Lie algebra which is isomorphic with respect to the Lie algebra of the group $U_{16} = U(2) \otimes U(2)$

$\otimes U(2) \otimes U(2)$. A transformation to x space makes it possible to show that U_{16} contains the transformations of the $U(1)$ subgroup (Heaviside-Larmor–Reinich[1,2]), of the $U(1) \otimes U(1)$ subgroup (Danilov[4] and Ibragimov[5]), and of the $U(2) \otimes U(2)$ subgroup (Fushchich and Nikitin[2]). Except for $U(1) \otimes U(1)$, all other transformations in x space turn out to be nonlocal (integral), in agreement with the conclusion reached in Ref. 2 regarding the subgroup $SU(2) \otimes SU(2)$.

[1]V. I. Strazhev and L. M. Tomil'chik, *Electrodynamics with Magnetic Charge*, Nauka Tekhnika, Minsk, 1975, p. 18.

[2]V. I. Fushchich and A. G. Nikitin, *Symmetry of Maxwell's Equations*, Naukova Dumka, Kiev, 1983, p. 5, 6, 23, 74.

[3]G. A. Kotel'nikov, in: *Group-Theory Methods in Physics*, Vol. 1, Nauka, Moscow, 1980, p. 203.

[4]Yu. A Danilov, Preprint 1452, I. V. Kurchatov Institute of Atomic Energy, Moscow, 1967, p. 18.

[5]N. Kh. Ibragimov, Dokl. Akad. Nauk SSR **178**, 566 (1968) [Sov. Phys. Dokl. **13**, 229 (1968)].

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