Duality and β functions in 2D Freedman-Townsend theory

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The metric and twisting potential are derived explicitly for dual versions of the 2D Freedman-Townsend theory and the SU(2) group. Computer calculations of two-loop β functions are reported.

Freedman and Townsend[1] proved the classical equivalence of a self-affecting antisymmetric gauge tensor field (the *FT* model) and the principal chiral σ model in $d=4$ by means of a dual transformation. It is pertinent to note that in the 2D case for the SU(2) group the *FT* model (a nonlinear σ model with twisting) and its dual principal chiral σ model can be constructed explicitly.

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We consider a Freedman-Townsend theory in $d=2$ with a first-order Lagrangian

$$L_1 = -\frac{1}{2} B_a \epsilon^{\mu\nu} F_{\mu\nu a} - \frac{1}{2} A_{\mu a} A^{\mu}_{a}, \tag{1}$$

where $F^a_{\mu\nu}$ is the non-Abelian strength of the gauge field A^a_μ for the SU(2) group with the structure constants ϵ^{abc}, where $a,b = 1,2,3$.

The equations of motion for $A_{\mu a}$ from action (1),

$$2\partial_\mu B^c \epsilon^{\mu\nu} + (\eta^{\mu\nu}\delta^{bc} + \epsilon^{\mu\nu} B_a \epsilon^{abc}) A^b_\mu = 0, \tag{2}$$

are algebraic with respect to A^b_μ and can be solved explicitly:

$$A^\mu_a = \frac{2}{1+B^2} (\epsilon^{\mu\nu} \partial_\nu B_a + \epsilon^{\mu\nu} B_a B_b \partial_\nu B_b + \epsilon_{abc} B_b \partial^\mu B_c), \quad B^2 \equiv B_a B_a. \tag{3}$$

This is a nontrivial property by virtue of the B dependence of the matrix with A^b_μ in (2).

After substituting (3) into (1), we find a nonlinear 2D σ model with a Wess-Zumino-Witten term (or twisting):

$$\frac{1}{8} L_{\sigma(1)} = -\frac{1}{2} [g_{ab}(B) \partial_\mu B_a \partial^\mu B_b + h_{ab}(B) \epsilon^{\mu\nu} \partial_\mu B_a \partial_\nu B_b], \tag{4}$$

where

$$g_{ab} = \frac{\delta_{ab} + B_a B_b}{1+B^2}, \quad h_{ab} = \frac{-\epsilon_{abc} B_c}{1+B^2}. \tag{5}$$

On the other hand, by varying (1) with respect to B_a, we find the constraint $F_{\mu\nu a} = 0$, which can be solved for A^a_μ,

$$A^a_\mu = 2M_b{}^a(\theta) \partial_\mu \theta^b, \tag{6}$$

in terms of scalar fields (local coordinates) θ^a and the local edge $M^a_b(\theta)$, which satisfies the Maurer-Cartan equations

$$\frac{\partial M_b{}^a}{\partial \theta^c} - \frac{\partial M_c{}^a}{\partial \theta^b} + 2\epsilon^{adf} M_c{}^d M_b{}^f = 0. \tag{7}$$

After substituting (6) into (1), we find the Lagrangian of the principal chiral σ model for SU(2), which is the dual of (4):

$$L_{\sigma(2)} = -\frac{1}{2} \hat{g}_{ab}(\theta) \partial_\mu \theta^a \partial^\mu \theta^b, \tag{8}$$

where the metric is

$$\hat{g}_{ab}(\theta) \equiv 4M_a{}^c(\theta)M_b{}^c(\theta). \tag{9}$$

To find the metric $\hat{g}_{ab}(\theta)$ explicitly, we use a formal solution of the Maurer-Cartan equations[2]:

$$M_{ab}(\theta) = \frac{1}{2}\,\mathrm{tr}\,(T_b U^{-1}\partial_a U) = -\frac{1}{2}\int_0^1 dt\,\mathrm{tr}\,(T_b U^{-t} T_a U^t), \tag{10}$$

where

$$U(\theta) \equiv \exp\,\{ i \sum_{a=1}^{3} \theta^a T_a \}, \tag{11}$$

and T_c are the generators of SU(2) in the associated representation. Using the easily verifiable identities

$$\begin{aligned} &(\theta^a T_a)^{2k+1} = R^{2k}\theta^a T_a, \\ &(\theta^a T_a)^{2k+2} = R^{2k}(\theta^a T_a)^2, \\ &\mathrm{tr}\,[T_a(\theta^b T_b)^2 T_c(\theta^f T_f)] = 0, \\ &\mathrm{tr}\,[T_a(\theta^b T_b)^2 T_c(\theta^f T_f)^2] = 2R^2\theta_a\theta_c, \end{aligned} \tag{12}$$

in which we have $R^2 \equiv \theta_1^2 + \theta_2^2 + \theta_3^2$, and which follow from elementary properties of the SU(2) generators,

$$\begin{aligned} &[T_a, T_b] = i\epsilon_{abc}T_c, \quad \mathrm{tr}\,(T_a T_b) = 2\delta_{ab}, \\ &\mathrm{tr}\,(T_a T_b T_c) = i\epsilon_{abc}, \quad \mathrm{tr}\,(T_a T_b T_c T_d) = \delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc}, \end{aligned} \tag{13}$$

we find

$$U^{\pm t} = 1 \pm \frac{i\sin Rt}{R}(\theta^a T_a) + \frac{\cos(Rt) - 1}{R^2}(\theta^a T_a)^2. \tag{14}$$

Substituting (14) into (10) and integrating, we find

$$M^{ab}(\theta) = -\frac{\sin R}{R}\delta^{ab} - \frac{\cos R - 1}{R^2}\epsilon^{abc}\theta^c - \frac{R - \sin R}{R^3}\theta^a\theta^b. \tag{15}$$

We thus find

$$\hat{g}_{ab}(\theta) = \frac{4(2\cos R - \cos 2R - 1)}{R^4}(R^2\delta_{ab} - \theta_a\theta_b) + \frac{4\theta_a\theta_b}{R^2}. \tag{16}$$

A renormalization of a nonlinear 2D σ model may be understood as a quantum deformation of the geometry of the field manifold. In turn, the renormalization is characterized by the renormalization-group β functions. To calculate specific values of the β functions from the given metric and the twisting potential, we have used a system for analytic calculations on a computer (the version REDUCE 3.0) and the general results.[3]

For the theory of (5), the single-loop β functions are

$$(2\pi)\beta_{ab}^{(1)} = \frac{1}{(1+B^2)^2}\{g_{ab}[\frac{1}{2}(3+B^2)^2 + B^2 - 3] - \delta_{ab}(3+B^2)\}, \tag{17}$$

$$(2\pi)\check{\beta}_{ab}^{(1)} = -\frac{2h_{ab}}{(1+B^2)^2},$$

in agreement with results of "manual" calculations.[1)]

For the dual theory (8) with metric (16) we find the result

$$(2\pi)\hat{\beta}_{ab}^{(1)} = \frac{K''+2}{2R^4}(\theta_a\theta_b - R^2\delta_{ab}) + (2KK'' - K'^2)\frac{\theta_a\theta_b}{2K^2R^2}, \tag{18}$$

where the prime means differentiation with respect to R, and

$$K \equiv \cos 2R - 2\cos R + 1. \tag{19}$$

Computer calculations of the single-loop β function of the σ model in (4)–(5) lead to the results

$$(4\pi)^2\beta_{ab}^{(2)} = \frac{1}{2}(1+B^2)^{-5}\{B_aB_b[-3(1+B^2)^4 - 24(1+B^2)^3 \tag{20}$$

$$-48(1+B^2)^2 + 48(1+B^2) - 32] + \delta_{ab}[-3(1+B^2)^4 + 80(1+B^2) - 32]\},$$

$$(4\pi)^2\tilde{\beta}_{ab}^{(2)} = (1+B^2)^{-4}h_{ab}\{(1+B^2)^2 + 4(1+B^2) - 2\}.$$

The corresponding result for model (16) is

$$(4\pi)^2\hat{\beta}_{ab}^{(2)} = \frac{-1}{16R^4K^3}[(2K''K - K'^2)^2 + (K'^2 + 4K)] \tag{21}$$

$$\times(\theta_a\theta_b - R^2\delta_{ab}) - \frac{1}{8R^2K^4}(2K''K - K'^2)\theta_a\theta_b.$$

Consequently, different geometries (different metrics and twistings) and different β functions correspond to different dual versions of the same classical 2D theory. Quantum equivalence does not preclude differences in the β functions, since the latter are determined with respect to different fields.

The basic results of this study are Eqs. (5), (16)–(18), (20), and (21), which are useful for applications.

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[2]E. Braaten *et al.*, Nucl. Phys. B **260**, 630 (1985).
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