Effective conductivity of a three-dimensional two-phase medium

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A randomly inhomogeneous, three-dimensional, two-phase medium is studied. An asymptotically exact expression for an effective conductivity at the percolation threshold is derived.

The problem of effective conductivity of highly inhomogeneous media has been solved exactly only in the two-dimensional case.[1] In the present letter we will consider a conducting medium which was obtained by randomly mixing two phases. We will show that at the percolation threshold (with equal phase concentrations) the effective conductivity of the system is equal to the geometric mean of the conductivities of the phases. The problem has not been solved for arbitrary concentrations.

Since there are no corresponding results in the three-dimensional case, a description based on the scaling hypothesis must be the principal approach. According to this hypothesis, the effective conductivity σ_e of a strongly inhomogeneous, two-phase system near the percolation threshold can be represented in a self-similar way[2]:

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$$\sigma_e = \sigma_1 h^s f(\epsilon/h^{s/t}), \tag{1}$$

where σ_1 is the conductivity of the first phase, the parameter $h = \sigma_2/\sigma_1$ is the ratio of the conductivities of the phases ($h \ll 1$), and ϵ is the deviation from the three-dimensional percolation threshold. The percolation occurs in the first phase. Asymptotic behavior of the function f is described by the indices of the percolation theory s, t, and q:

$$f(z) = \begin{cases} |z|^{-q}, & z \ll -1 \\ 1, & |z| \ll 1. \\ z^t, & z \gg 1 \end{cases} \tag{2}$$

These indices are related by the relation

$$t(1/s - 1) = q. \tag{3}$$

We note that a scaling description of conductivity is a standard description which has been confirmed by numerical simulation.

Let us consider an effective conductivity of a highly inhomogeneous, three-dimensional, two-phase system at the percolation threshold ($\epsilon = 0$), where the conductivity is characterized exclusively by the index s. The phases are randomly distributed, as in the two-dimensional model. Our objective is to determine the exact value of the critical index s. We will show below that the value of this index is 2/3, i.e.,

$$\sigma_e \sim \sigma_1^{1/3} \sigma_2^{2/3} \qquad \text{for} \quad \sigma_2 \ll \sigma_1. \tag{4}$$

The method used to determine the index can be described as follows. We will proceed from the fact that the dependence of the transverse conductivity σ_{xx}^e on the magnetic field **H** at the percolation threshold is known. According to Ref. 3, this dependence is described by the four-thirds power law:

$$\sigma_{xx}^e \sim \sigma/(\beta)^{4/3}, \text{ where } \quad \vec{\beta} = \frac{e\mathbf{H}}{mc} \tau. \tag{5}$$

We will show that at the percolation threshold this dependence of the conductivity σ_{xx}^e on the magnetic field is described by the index s; specifically,

$$\sigma_{xx}^e \sim \sigma/(\beta)^{2s}. \tag{6}$$

Accordingly, we obtain from (5) and (6) an exact expression for the index s, and thus confirm expression (4).

We will express the dependence of the transverse conductivity on the magnetic field in terms of s. We will accordingly establish a relationship between the conductivity and the galvanomagnetic properties. (The possibility of establishing such a relationship was initially suggested by Balagurov.[4]) We will employ the arguments used by Dykhne.[5] The direct-current equations which describe a comductor

$$\operatorname{div} \mathbf{j} = 0 \qquad \operatorname{curl} \mathbf{e} = 0 \tag{7}$$

and the generalized Ohm's law in a magnetic field

$$\mathbf{j} + [\mathbf{j}\vec{\beta}] = \sigma\mathbf{e} \tag{8}$$

are unaffected by the linear transformations

$$\mathbf{j} = a\mathbf{j}' + b[\mathbf{n}\mathbf{e}'], \quad \mathbf{e} = c\mathbf{e}' + d[\mathbf{n}\mathbf{j}'], \tag{9}$$

where $\mathbf{n}$ is a unit vector directed along the magnetic field, and a, b, c, and d are constants.

In the three-dimensional case it is necessary to use the truncated transformations: transformations (9) with the coefficient $d = 0$. For the characteristics of the transformed (primed) system we have

$$\sigma' = \sigma\frac{c}{a} - 2\beta\frac{b}{a} + \frac{b^2(1+\beta^2)}{\sigma ac}, \quad \beta' = \beta - \frac{b(1+\beta^2)}{\sigma c}. \tag{10}$$

Similar expressions can be obtained for the effective values, and the primed system is chosen on the basis of the secondary arguments.

Let us assume that the original system is characterized exclusively by the conductivities σ_1 and σ_2 of the phases ($\beta_1 = \beta_2 = 0$). As the primed system we chose the system

$$\sigma_1' = \sigma_2' = \sqrt{\sigma_1\sigma_2}. \tag{11}$$

The Hall components of the primed system are determined uniquely from (10):

$$\beta_1' = -b/c\sigma_1 \qquad \beta_2' = -b/c\sigma_2. \tag{12}$$

The transformation which determines the exact relationship between the indicated systems at arbitrary phase concentrations is given by the coefficients

$$\sigma_e' = \frac{c}{a}\sigma_e + \frac{b^2}{ca\sigma_e}, \quad \beta_e' = -\frac{b}{c\sigma_e}. \tag{13}$$

We accordingly obtain a relationship between the effective characteristics of the original two-phase system and the primed system:

$$a = 1, \quad b/c = \sqrt{\sigma_1\sigma_2}, \quad c/a = \frac{\sqrt{\sigma_1\sigma_2}}{\sigma_1 + \sigma_2}. \tag{14}$$

We wish to emphasize that relations (14) hold for any relationship between the conductivities and any phase concentration.

Let us analyze expression (14) for a highly inhomogeneous medium at the percolation threshold, where the effective conductivity of the original system is $\sigma_e = \sigma_1 h^s (h \ll 1)$. In this limit the effective values of the primed system, according to (14), are

$$\sigma_e' \sim \sigma'/(\beta_2')^{2(1-s)}, \quad \beta_e' \sim \beta_2'/(\beta_2')^{2(1-s)}. \tag{15}$$

Here $\sigma^1 = \sqrt{\sigma_1 \sigma_2}$, and $\beta'_2 = -\sqrt{\sigma_1/\sigma_2}$.

Expressions (15) hold in both the two-dimensional and three-dimensional case. Only the critical index s depends on the dimensionality of space. In the 2D case expressions (15) become, as expected, the corresponding expressions of Ref. 5. Expressing the conductivity tensor σ_{xx} in terms of σ and β in the standard way: $\sigma_{xx} = \sigma/(1+\beta^2)$, we find expression (6).

In conclusion, let us discuss the four-thirds square law. Strictly speaking, this dependence was found for the cases $\beta_1 \gg 1$ and $\beta_2 \gg 1$. From Ref. 3, however, we can find the same dependence in the interval of interest to us: $\beta_1 \ll 1 \ll \beta_2$. Using the method of Ref. 3, this limit was considered in Refs. 6 and 7, and the four-thirds square law was obtained (although Ref. 3 was not cited in Ref. 6). Let us discuss the universality of this law. As was shown in Ref. 3, anomalous magnetoresistance (5) is a consequence of the change in the nature of transverse diffusion in strong magnetic fields. This mechanism requires that the Hall concentrations in the different phases must differ considerably. This condition holds in our case. The two-dimensional problem on magnetoresistance is solved exactly in the case of arbitrary values of β_1 and β_2 (Ref. 5). In each limiting case: $\beta_1 \ll 1 \ll \beta_2$ and $\beta_1 \gg 1$, $\beta_2 \gg 1$, we obtain the same magnetic-field dependence of σ^e_{xx}.

The value $s = 2/3$ is also in good agreement with the results of numerical simulation: 0.62.

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