Non-Abelian Einstein-Yang–Mills black holes

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(Submitted 7 September 1989)
Pis'ma Zh. Eksp. Teor. Fiz. **50**, No. 7, 312–315 (10 October 1989)

Solutions of the self-consistent system of Einstein-Yang–Mills equations with the SU(2) group are derived to describe black holes with a non-Abelian structure of gauge fields in the external region.

In the case of the electrovacuum, the most general family of solutions describing spherically symmetric black holes is the two-parameter Reissner–Nordström family, which is characterized by a mass M and an electric charge Q. It was recently shown for the Einstein-Yang–Mills systems of equations with the SU(2) group that a corresponding assertion holds when the hold has a nonvanishing color-magnetic charge. In this case the structure of the Yang–Mills hair is effectively Abelian.[1] In the present letter we numerically construct a family of definitely non-Abelian solutions for Einstein-Yang–Mills black holes in the case of zero magnetic charge. These solutions are characterized by metrics which asymptotically approach the Schwarzschild metric far from the horizon but are otherwise distinct from metrics of the Reissner–Nordström family. In addition to the complete Schwarzschild metric, the family of solutions is parametrized by a discrete value of n: the number of nodes of the gauge function. For a given value of the radius of the event horizon, the mass of the solution increases with increasing n. At $n = 0$ the solution is of a Schwarzschild nature everywhere, and the Yang-Mills field is identically zero. The solutions for $n \neq 0$ are new and have no analogs in the cases of the vacuum and the electrovacuum.

We use the following representation of the space-time interval and of the 1-forms of the Yang–Mills field of a static, spherically symmetric configuration with the SU(2) group:

$$ds^2 = \frac{\Delta(r)}{r^2}\sigma^2(r)dt^2 - \frac{r^2}{\Delta(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

$$A^a_\mu dx^\mu \frac{\tau^a}{2} = \frac{(f(r)-1)}{g}(\mathbf{e}_\varphi d\theta - \sin\theta\, \mathbf{e}_\theta d\varphi)\frac{\vec{\tau}}{2}, \tag{2}$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are unit vectors of the spherical coordinate system; $\vec{\tau}$ are the Pauli matrices; $\Delta(r)$, $\sigma(r)$, and $f(r)$ are functions of the radial variable which are to be determined; and g is the gauge coupling constant. The family of solutions in which we are interested is specified by the asymptotic conditions as $r \to \infty$:

$$\frac{\Delta}{r^2} \to 1, \quad \sigma \to 1, \quad f \to \pm 1 \tag{3}$$

These conditions follow from the assumptions that the metric is asymptotically planar, that there is no magnetic charge, and that an event horizon exists:

$$\exists r_H > 0: \quad \Delta(r_H) = 0, \qquad \Delta(r) > 0 \quad \text{for} \quad r > r_H \,. \tag{4}$$

Eliminating the function σ from the system of Einstein-Yang–Mills equations,[1] we find a system of two equations for Δ and f:

$$Df'' + \left(1 - \alpha \frac{(f^2-1)^2}{4x}\right) f' = \frac{f(f^2-1)}{2x} \tag{5a}$$

$$D' + \left(\alpha f'^2 - \frac{1}{2x}\right) D = 1 - \alpha \frac{(f^2-1)^2}{4x}, \tag{5b}$$

where $x=(r/r_H)^2$, $D=2\Delta/r_H^2$, $\alpha=(16\pi/g^2)(l_p/r_H)^2 \equiv (R_*/r_H)^2$ and l_p is the Planck length. The function σ is expressed in terms of the solution $f(x)$:

$$\sigma(x) = \exp\left(-\alpha \int_x^\infty f'^2 dx\right). \tag{6}$$

System of equations (5a), (5b) has been solved numerically by the Runge–Kutta method with initial data on the $x=1$ event horizon. It can be shown that for a given α the set of initial data is parametrized by the single quantity $f^* \in [-1,0]$:

$$D(1) = 0, \quad f(1) = f^*, \qquad f'(1) = f'(f^*). \tag{7}$$

It turns out that solutions which satisfy asymptotic expression (3) correspond to the discrete set $f_n^*(\alpha)$, where $n=0,1,2,\ldots$. In the case $n=0$ we have $f\equiv 1$, and the metric is of a Schwarzschild nature everywhere. Even values of n correspond to the asymptotic value $f=-1$, and odd values to $f=1$. The behavior of the solution $f(x)$ with $f(1)=f_n^*(\alpha)$, $n=1,2,\ldots$, can be described qualitatively as follows: The function f increases near the horizon, oscillates, crossing zero n times, and then asymptotically

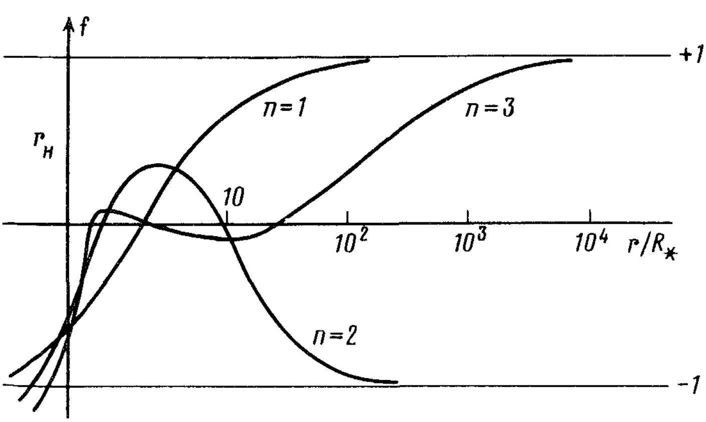

FIG. 1. Solutions for the function $f(r)$ for $\alpha=4$.

approaches the value $(-1)^{n+1}$. We have $|f(x)| < 1$ over the entire interval $1 \leqslant x < \infty$, which corresponds to the external region of the hole (Fig. 1).

The metric function Δ can be written in the form $\Delta = r_H^2 (x - 2\sqrt{x} m(x))$, where the dimensionless mass function $m(x)$ increases monotonically from $m_0 \equiv m(1) = 1/2$ at the horizon to $m(\infty) = m_0 + m_n(\alpha)$. The Schwarzschild mass of the solution is

$$M_n(\alpha) = \frac{m(\infty)}{\sqrt{\alpha}} M_* , \qquad (8)$$

where $M_* = (\sqrt{16\pi/g^2}) m_p$, and m_p is the Planck mass. The total mass of the black hole is thus represented as the sum of a seed mass below the horizon and a field mass which is distributed in the external region of the hole. The function σ also increases monotonically from a certain value $\sigma_n^*(\alpha) < 1$ at the horizon to $\sigma = 1$ at infinity. In the limit $\alpha \to 0$ we have $m_n(\alpha) \to 0$ and $\sigma_n^*(\alpha) \to 1$, and in the external region the metric is essentially the same as the Schwarzschild metric produced by the mass $M_*/2\sqrt{\alpha}$ (which tends toward infinity). At nonzero values $\alpha \gtrsim 1$ the function Δ can be approximated quite accurately near the horizon by the Reissner–Nordström expression $\Delta = r^2 - 2M_n(\alpha) r + Q^2$, where $Q^2 \approx (1/4) R_*^2$. As n increases, the size of this neighborhood also increases. In general, the value of Q^2 depends on r: at r above a certain $\bar{r}$ (which increases with increasing n), Q^2 decays rapidly to zero, so the metric turns out to be of a Schwarzschild nature asymptotically. For the Schwarzschild mass for each value of α we can find an empirical n dependence in this case. For $\alpha = 4$, for example, we find

$$M_n(4) = (0.5 - 0.369 \exp(-2.395 n)) M_* . \qquad (9)$$

A continuation of the solutions below the event horizon shows that there is at least one interior horizon. At large values of α the behavior of the solutions depends only

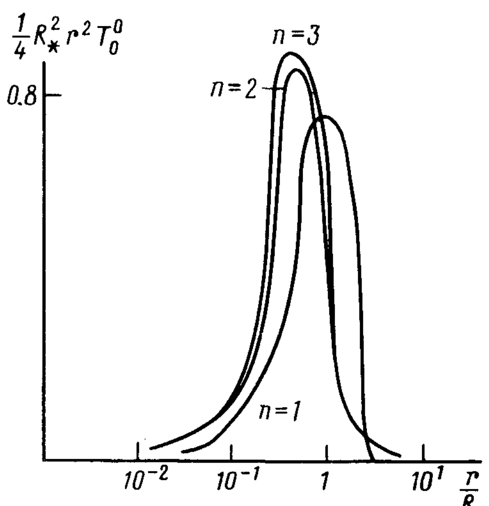

FIG. 2. Energy density of the external field configuration of a black hole for $\alpha = 10^6$.

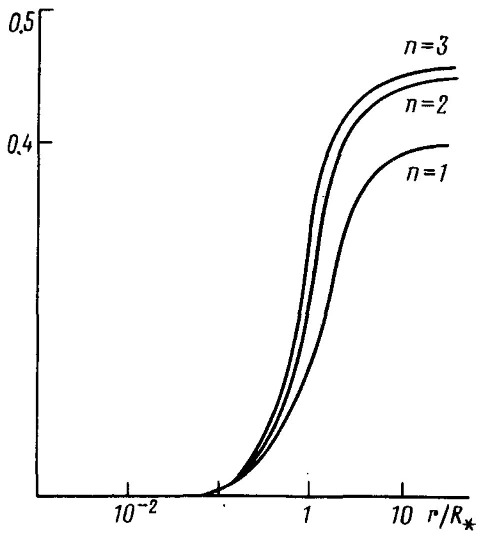

FIG. 3. Behavior of the mass function $m(r) = [m(r)/\sqrt{\alpha}]M_*$ for $\alpha = 10^6$.

weakly on α (beginning at values $\approx 10^3$). In the limit $\alpha \gtrsim 10^3$ one can distinguish three characteristic regions:

$$r_H = R_*/\sqrt{\alpha} \leqslant r \lesssim 0.1R_*(1), \quad 0.1R_* \lesssim r \lesssim R_*$$

and $R_* \lesssim r$ (III). In region I all the functions vary slowly ($f \approx f^* \approx -1$, $m \approx m_0$, $\sigma \approx \sigma^*$), and the metric is approximately the Schwarzschild metric corresponding to the mass m_0, with an additional slowing of the coordinate time: $t \to t' = \sigma^* t$. As $\alpha \to \infty$ we have $1/\sigma_n^*(\infty) = 7.9$, 48.1, and 297.6 for $n = 1, 2, 3$, respectively. In region II the

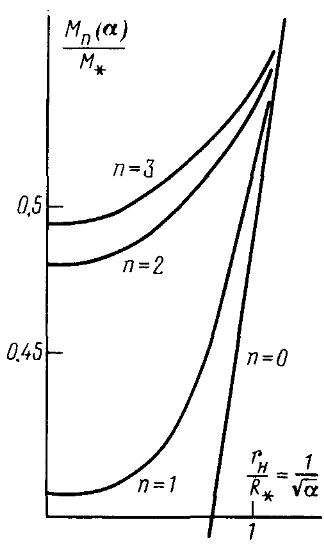

FIG. 4. Schwarzschild mass of the black hole, $M_n(\alpha)$, versus the parameters α and n of the solution.

function f oscillates around zero, the energy density $m'(r)$ has a sharp maximum at $r \approx 0.4 R_*$ (Fig. 2), and the functions m and σ increase rapidly to their asymptotic values (Fig. 3). Nearly all the energy of the solution is concentrated in this region. In region III the metric is of a Schwarzschild nature, and the function f approaches its asymptotic value $(-1)^{n+1}$. In the limit $\alpha \to \infty$ the Schwarzschild mass of the solutions is $M_n(\infty)/M_* = 0.414$, 0.486, and 0.498 for $n = 1$, 2, and 3, respectively. In the interior region of the black hole at large values of α the deviation of the metric from a Schwarzschild metric is on the order of $1/\alpha$, and it tends towards zero as $r \to 0$. As $\alpha \to \infty$, the horizon radius and the seed mass of the solutions tend toward zero as $1/\sqrt{\alpha}$, so region I extends all the way to the singularity at $r = 0$. In this region the metric is a Schwarzschild metric with a vanishingly small mass. Interestingly, the complete masses of the solutions found in this limit differ [they are smaller by a factor of about two) from the corresponding masses of the regular field configuration described in Ref. 2 for which the metric is planar at $r = 0$. This distinction stems from the difference between the boundary conditions at the origin of coordinates.

In summary, black holes corresponding to solutions of the self-consistent system of Einstein-Yang–Mills equations form a wider family than in the cases of the vacuum and the elecrovacuum, which have been studied previously. This circumstance should be kept in mind in a discussion of the hypothesis of primeval black holes, formed in an early stage of the cosmological expansion, before the time of the phase transition which distinguished electromagnetic interactions. The solutions described here are not discriminated against by the existing theorems which state that there is no hair, since the external field configuration of black hole is in a state of a nonradiative multipole. Its existence does not, however, lead to the appearance of gauge charges of the hole because of the rapid decay of the fields at infinity.

[1]D. V. Gal'tsov and A. A. Ershov, Phys. Lett. A **138**, 160 (1989).
[2]R. Bartnik and J. McKinnon, Phys. Ref. Lett. **61**, 141 (1988).

Translated by Dave Parsons