Central expansion of Lie algebra of differential operators on a circle and W algebras

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(Submitted 12 September 1989)
Pis'ma Zh. Eksp. Teor. Fiz. **50**, No. 8, 341–343 (25 October 1989)

1. We consider the affine space

$$\mathfrak{L} = \{L = \partial_x^n + u_{n-1}\partial_x^{n-1} + \ldots + u_0\}$$

of differential operators of order n on the circle S^1 with a highest-order coefficient of unit. We assume that the coefficients of the operators are smooth functions on the circle. The space which is tangent to $\mathfrak{L}$ consists of differential operators of order no higher than $n-1$. We recall[1,2] that a "Gel'fand–Dikiĭ algebra" is a Lie subalgebra of vector fields on $\mathfrak{L}$ of the type

$$V_X(L) = L(XL)_+ - (LX)_+L. \tag{1}$$

Here $X = \partial_x^{-1} \circ x_{-1} + \partial_x^{-2} \circ x_{-2} + \ldots$ is a pseudodifferential symbol, $(\Sigma_{-\infty}^n a_i \partial_x^i)_+ = \Sigma_0^N a_i \partial_x^i$. means to take the differential part of a symbol, $A_- = A - A_+$ means to take the integral part of a symbol, and the coefficients x_{-i} are differential polynomials of the functions u_i, i.e., elements of the ring $k[u_i^{(j)}]$ [here k is the field R or C, and $u_i^{(j)} = \partial_x^j(u_i)$]. The vector field $V_X(L)$ is essentially independent of the coefficients X with indices lower than n. The commutator of vector fields on $\mathfrak{L}$ is determined by, say, the action of a vector field on the "functions" (i.e., on the functional $F[L]$ of the functions u_j, which are the coefficients of operator L)

$$V_x(L)(F) = \frac{d}{d\epsilon}F[L + \epsilon V_x(F)]|_{\epsilon=0}.$$

A Gel'fand–Dikiĭ algebra of course includes the Lie algebra of vector fields on a circle (Ref. 2, for example). It turns out that the following assertion holds.

A *Gel'fand–Dikiĭ* algebra contains a Lie algebra of differential operators on a circle.

To demonstrate this assertion, we associate with the differential operator

$$E = e_0 + e_1\partial_x + e_2\partial_x^2 + \ldots + e_p\partial_x^p$$

the vector field

$$W_E(L) = LE - (LEL^{-1})_+L = (LEL^{-1})_-L. \tag{2}$$

We obviously have

$$W_E(L) = V_{(EL^{-1})_-}(L). \tag{3}$$

Direct calculations verify that we have

$$[[W_E(L), W_F(L)]] = W_{[E,F]}(L). \quad (4)$$

Standing on the left side of (4) is the commutator of vector fields on $\mathfrak{L}$.

Comment 1. We wish to stress that the differential operators E and F in (2), (3), and (4) do not depend on L. The corresponding assertion would not be correct in a Gel'fand–Dikiĭ algebra, as we know quite well: If $[[V_X(L), V_Y(L)]] = V_Z(L)$ for L-independent X and Y, then Z will generally become dependent on L.

2. Relation (2) can be derived in the following way. We consider the solutions f of the equation $Lf = 0$. We assume that the differential operators E act in an infinitesimal fashion on these solutions, sending them into the solutions of another n-th-order differential equation:

$$(L + \epsilon\Lambda)(f + \epsilon Ef) = 0 \mod \epsilon^2.$$

We then have $\Lambda = -W_E(L)$, and now relation (4) becomes obvious.

3. It is of course understandable that only differential operators of order no higher than $n-1$ will effectively act on the solutions of n-th order equations, and the effect of higher-order operators reduces to the effect of an operator of an order no higher than $n-1$, which, of course, depends on L:

$$Ef \doteq (E - (EL^{-1})_+ L)f. \quad (5)$$

2. We now change the subject. The following assertion holds. *A Lie algebra of differential operators on a circle allows a nontrivial centrtal expansion by means of numbers*:

$$0 \to k \to DOP^{\wedge}(S^1) \to DOP(S^1) \to 0. \quad (6)$$

The corresponding 2-cocycle can be specified by

$$c(f_m \partial_x^m, g_n \partial_x^n) = \frac{m!\,n!}{(m+n+1)!} \int_{S^1} f_m^{(n)} g_n^{(m+1)} dx. \quad (7)$$

That (7) determines a cocycle can be verified by direct calculation. In the process we use the following identities with binomial coefficients:

$$\sum_{k=0}^{n} \frac{(-1)^k (m+n-k)!n!}{(m+n+p-k)!k!(n-k)!} = (-1)^n \frac{m!(n+p-1)!}{(p-1)!(m+n+p)!} \text{ for } p \geqslant 1$$

$$\sum_{k=0}^{n} \frac{(-1)^k (m+n+p-k)!n!}{(m+n-k)!k!(n-k)!} = \frac{p!(m+p)!}{(p-n)!(m+n)!} \text{ for } p \geqslant n, \quad 0 \text{ for } p < n.$$

That the cocycle is nontrivial follows from, for example, the circumstance that when there is a limitation on the Lie algebra of vector fields, it converts into the Gel'fand–Fuks cocycle (Ref. 3, for example).

Comment. According to Ref. 4, the Lie algebra $DOP(S^1)$ has a unique nontrivial central expansion by means of numbers.

3. In conclusion we would like to point out the following: As Luk'yanov has shown,[5] the Gel'fand–Dikiĭ Lie algebras is the classical limit of the so-called W_n algebra,[6–8] which contains, along with the energy-momentum tensor—a conformal-symmetry generator—chiral currents of spin n. The geometric meaning of W_n symmetry remains unclear.

Now, in accordance with the discussion in *1, we have found a basis for the suggestion that the classical limit of the W_n algebra is the transform of the Lie algebra of differential operators on a circle during application to the solutions of n-th-order differential equations. We see that the W algebras themselves are related to a corresponding factorization of central expansion (6) of the Lie algebra of differential operators. This expansion plays the role of a "universal W algebra." Universal W algebras have been studied independently by Morozov.[9]

It is extremely likely that cocycle (7) can be generalized to a Lie superalgebra of differential operators which correspond to Lie superalgebras of string theories.[10]

Incidentally, a recent preprint[11] dealt with related questions.

I wish to thank A. Gerasimov, S. Luk'yanov, A. Marchakov, A. Morozov, A. Perelomov, and T. Khovanova for stimulating discussions. I also thank M. Khovanov for calling Ref. 4 to my attention.

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Translated by Dave Parsons