Quantization on sphere and high-spin superalgebras

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The relationship between the algebras of quantum operators on a 2D sphere and on a hyperboloid with infinite-dimensional high-spin algebras is established. The quantum-deformation parameter (Planck's constant) is expressed in terms of the expectation value of one of the scalar fields of a high-spin theory.

The role played by infinite-dimensional symmetries in relativistic field theory increases in importance with every passing year. The Virasoro and Kac–Moody algebras, which arose in the context of string theory, have recently been joined with the algebras of simplectic diffeomorphisms (Poisson brackets) of 2D manifolds in membrane theory[1] and infinite-dimensional algebras associated with high-spin gauge theories.[2–6] It has been established[7,8] that the simplest versions of the high-spin algebras which were discussed in Refs. 2, 3, and 6 can be interpreted as the result of a quantization of a Poisson-brackets algebra of a 2D hyperboloid with a fixed value of "Planck's constant" $\hbar$. Our basic purpose in the present letter is to offer analogous interpretation for quantum algebras on a hyperboloid and on a sphere for arbitrary values of $\hbar$ in some half-interval. The construction proposed below turns out to be an effective technical facility, which makes it possible, in particular, to construct in a simple way invariant quadratic forms of corresponding Lie (super-) algebras. At the same time, it leads to an unexpected physical interpretation of quantum algebras on a sphere and a hyperboloid: Planck's constant is expressed in terms of the expectation value of one of the auxiliary scalar fields of Ref. 5. The interrelationship between quantization effects and effects of a spontaneous symmetry breaking which arises in high-spin theories as a result is extremely intriguing.

The construction can be summarized as follows. We introduce operators q_α ($\alpha = 1,2$) and Q which, by definition, satisfy the relations

$$Qq_\alpha = -q_\alpha Q, \quad Q^2 = I, \quad [q_\alpha, \ q_\beta] = 2i\epsilon_{\alpha\beta}(1+\upsilon Q), \tag{1}$$

where υ is an arbitrary numerical parameter ($\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \varepsilon_{12} = 1$). We consider the associative algebra $Aq(2;\upsilon)$ of all possible polynomials

$$P(Q,q) = \sum_{\substack{n=0 \\ A=0,1}}^{\infty} \frac{1}{n!} f^{A\alpha_1 \ldots \alpha_n} (Q)^A q_{\alpha_1} \ldots q_{\alpha_n} \tag{2}$$

(The numerical coefficients $f^{A\alpha_1 \cdots \alpha_n}$ are assumed to be completely symmetric in the indices $\alpha_1 \cdots \alpha_n$; this situation corresponds to a Weyl arrangement of the operators q_α.) The algebra $Aq(2;\upsilon)$ contains the subalgebra of even polynomials, $P(Q,-q) = p(Q,q)$, which in turn can be broken up into the sum of two algebras,

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$Aq^E(2;v) \oplus Aq^E(2;-v)$, formed by the elements $\Pi_\pm P_\pm(q)$, where $\Pi_\pm = 1/2(1+Q)$ and $P_\pm(-q) = P_\pm(q)$.

We assume

$$x_k = \frac{a}{4}\eta_k^{\alpha\beta} q_\alpha q_\beta , \tag{3}$$

where $a \neq 0$ is a real parameter and where $\eta_0^{\alpha\beta}$ and $\eta_{1,2}^{\alpha\beta}$ are respectively the unit matrix and the symmetric Pauli matrices $\sigma_{1,3}$. Using (1), we can easily show that we have

$$[x_k, q^\alpha] = ia\eta_k^{\alpha\gamma} q_\gamma , \qquad [x_k, x_l] = 2ia\epsilon_{kln} x^n, \tag{4}$$

where $q^\alpha = \varepsilon^{\alpha\beta} q_\beta$, and the vector indices k, l, $n = 0, 1, 2$ are rotated by the $2+1$ Minkowski metric $(+--)$. As a result, regardless of the value of v, the x_k have the properties of generators $o(2,1) \sim \mathrm{sp}(2)$, and the q_α form a spinor representation of $o(2,1)$. Furthermore, a simple calculation yields $x_k x^k = \frac{1}{4}a^2(v^2 - 2vQ - 3)$. Projecting all the relations onto $Aq^E(2;v)$ with the help of Π_+, we find

$$x_k x^k = \frac{a^2}{4}(v+1)(v-3), \tag{5}$$

In other words, the x_k can be interpreted as the coordinates of a 2D hyperboloid of radius $R^2 = (a^2/4)(v+1)(v-3)$, quantized with a Planck's constant $\hbar = 2a$, and the algebra $Aq^E(2;v)$ can be interpreted as an algebra of operators constructed from x_k, i.e., as an algebra of quantum operators on a hyperboloid $S^{1,1}$. An important parameter of the problem is the dimensionless ratio $R^2/\alpha^2 = 1/4(v+1)(v-3)$. The superalgebras of high spins which were discussed in Refs. 7 and 8 correspond to the case $v=0$, i.e., $R^2/\alpha^2 = -3/4$. The realization proposed here describes all quantum algebras on $S^{1,1}$ with $R^2/\alpha^2 \geqslant -1$.

On the algebras $Aq(2;v)$ there exists a supertrace operation, which is given for any polynomial (2) by the relation

$$\mathrm{str}(P) = f^0 - v f^1 , \tag{6}$$

where f^0 and f^1 are the coefficients of zero power in q. This operation has the usual property

$$\mathrm{str}(P_1P_2) = -(-1)^{(\pi(P_1)+1)(\pi(P_2)+1)} \mathrm{str}(P_2P_1) , \tag{7}$$

if the parity is specified by the relation

$$P(Q,-q) = (-1)^{\pi(P)} P(Q,q), \tag{8}$$

which corresponds to a normal relationship between the spin and the statistics. The operation str makes it possible to construct invariant polylinear forms of Lie superalgebras, defined in terms of anticommutators in the algebra $Aq(2;v)$ and parity (8), as traces of products. The invariant bilinear form $(P_1,P_2) = \mathrm{str}(P_1P_2)$ is given by

$$(P_1, P_2) = \sum_{\substack{n=0 \\ A,B=0,1}}^{\infty} \frac{1}{n!} \alpha^{AB}(n) f_1^A{}_{\alpha_1 \dots \alpha_n} f_2^{B\alpha_1 \dots \alpha_n}, \qquad (9)$$

$$\alpha^{AB}(n) = i^n \prod_{l=0}^{[n/2]} \left(1 - \frac{v^2}{(2l+1)^2}\right)\left[\delta(A+B) + \frac{1}{2}(1+(-1)^n)\frac{v}{n+1}\delta(A+B+1)\right] \qquad (10)$$

for $n>0$ and $\alpha^{AB}(0) = \delta(A+B)$ {here $[m/2]$ means the greatest integer in $m/2$; the indices A and B are added modulo 2 $(1+1=0)$; $\delta(0)=1$; $\delta(1)=0$}.

It follows from (10) that form (9) degenerates if and only if $v=2k+1$ for some integer k. It turns out that in such cases the complex factor of the algebra $Aq(2;2k+1)/I_k$, where I_k is a two-sided ideal formed by null vectors of form (9), is the same as the algebra of the matrices $(2k+1)\times(2k+1)$. As a result, we find an unexpected interpretation of finite-dimensional associative matrix algebras (and, after a switch to anticommutators, of the classical Lie superalgebras associated with them), as particular cases of infinite-dimensional algebras $Aq(2;v)$ with odd v. This interpretation becomes exact for the expressions inside the supertrace sign (for example, the corresponding Chern–Simons forms are the same).

In a discussion of a quantum algebra on a hyperboloid it has been implicitly assumed[1] that we have $q_\alpha^\dagger = q_\alpha, Q^\dagger = Q$, and thus $x_k^\dagger = x_k$ (the parameter v is real). The real infinite-dimensional Lie superalgebra $h(2;v)$ which corresponds to these Hermitian conditions may be thought of as a new version of a high-spin superalgebra in $2+1$ dimensions, distinct from that proposed in Ref. 6. The circumstance that this new version contains in a natural way the anti-de Sitter $d=2+1$ algebra $o_+(2,1)\oplus o_-(2,1)$, whose terms are singled out by the projection operators[2] $\Pi_\pm$, should be counted as an advantage of this new version.

The case of a sphere corresponds to the Hermitian conditions $q_\alpha^\dagger = Qq^\alpha, Q^\dagger = Q$. Defining the coordinates of the sphere by means of the relation $x_k = (a/4)iq_\alpha q_\beta\, \sigma_k^{\alpha\beta}$, where $\sigma_{k\alpha}^\beta$ are the Pauli matrices $(k=1,2,3)$, we find

$$x_k^\dagger = x_k, \quad [x_k, x_l] = 2i\epsilon_{kln}\, x^n, \quad x_k x^k = -\frac{a^2}{4}(Qv+1)(Qv-3) \qquad (11)$$

[the indices are rotated by the metric $(+\,+\,+)$]. Expanding the algebra of polynomials from x_k and Q in a sum of two subalgebras by means of $\Pi_\pm$, we find $x_k^2 = -(a^2/4)(v+1)(v-3)$. In other words, by using this method we can describe algebras with $x_k^2/a^2 \leqslant 1$ (furthermore, we conclude that with $v\leqslant -1$ and $v\geqslant 3$ the algebra under consideration does not allow unitary representations, since x_k^2 becomes negative).

We conclude by pointing out one more possibility: that of taking two mutually commutative sets of algebras (q_α, Q) and $(r_{\dot\alpha} R)$, with mutually conjugate parameters v and $\bar v$, by setting $(q_\alpha)^\dagger = r_{\dot\alpha}, Q^\dagger = R$. This model corresponds to a complex sphere with a Lorentz algebra $o(3;C)\sim o(3,1)$ as a symmetry algebra. With $v=0$, the corresponding Lie superalgebra is the same as the superalgebra of high spins and auxiliary fields[4] which was used in Ref. 5 to formulate equations of motion of gauge high spins in $d = 3 + 1$. Note that the parameter v in (1) can be interpreted as the expectation value of the auxiliary field $C^{00}(0,0)$ of Ref. 5. This interpretation indicates a profound relationship between the problem of high-spin gauge fields and the geometry of 2D manifolds.

[1]The Hermitian conditions written below are meaningful not for associative algebras but for the Lie superalgebras associated with them, in the same way that the ordinary Hermitian conditions which distinguish the Lie algebras $u(n)$ cannot be imposed at the level of an associative algebra of matrices.

[2]Lorentz connectedness corresponds to the diagonal subalgebra $o_L(2,1)$, and the 1-form of the tetrad lies in $o_+(2,1) \oplus o_-(2,1)/o_L(2,1)$. The algebra $h(2;0)$ differs from the algebras discussed in Ref. 6 by virtue of its fermion sector, which corresponds to fields of an auxiliary type which were described in Ref. 9 for the case $d = 3 + 1$. We wish to stress that the question of what is to be called a "massless high-spin field" in $d = 2 + 1$ has no clear physical meaning, since such fields do not have intrinsic degrees of freedom.

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