The gravitational–topological Chern–Simons term in a film of superfluid ^{3}H-*A*

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Quantization of the hydrodynamic parameter, which has the meaning of a gravitational–topological charge in the Chern–Simons term by analogy with quantum field theory, is found in a ^{3}H-*A* film.

Some physical parameters in the effective hydrodynamic action of a quantum many-body system take on quantized values due to the nontrivial internal topological structure of the system. In quantum theory in a $(2+1)$-measurement these are the coefficients (charges) of the topological Chern–Simons term (see Ref. 1 and the cited references). In condensed media this kind of quantization arises for such parameters as the Hall conductivity σ_{xy} in two-dimensional electronic systems[2] and the coefficient of the Hopf invariant in the hydrodynamic action for the spin dynamics of magnetic systems, which determines the type of quantum statistics of the magnetic solitons.[3] The latter coefficient was calculated for a ^{3}H-*A* film[4] and for an antiferromagnet with definite symmetry type.[5] Other physical systems are also being examined for possible quantization of parameters. One of the candidates is quantum spin liquids with spatial and temporal parity violation (see e.g., Ref. 6).

Here we consider the quantization arising in the hydrodynamic action in the orbital and superfluid dynamics of a ^{3}H-*A* film. The previously considered term in the orbital action, which describes the internal quantum Hall effect in ^{3}He-*A* (Ref. 7), leads only to approximate quantization of the Hall parameter σ_{xy} in the limit when the superfluid gap $\Delta \ll \varepsilon_F$—the Fermi energy (see Ref. 7). It turns out that there is another term in the orbital action whose coefficient is quantized exactly. This term is analogous to the gravitational Chern–Simons term in quantum field theory[1]:

$$S = \frac{q_{gr}}{24\pi} \int d^2x dt e^{\mu\nu\lambda} (R_{\mu\nu,ab} \omega^{ab}_{\lambda} + \frac{2}{3} \omega^b_{\mu a} \omega^c_{\nu b} \omega^a_{\lambda c}), \tag{1}$$

where $\mu = (0,1,2)$ are the spatial indices, $a = (0,1,2)$ are the isotopic indices $\omega_{\mu,ab}$ is the spin coherence, and R is the curvature tensor

$$R_{\mu\nu,ab} = \partial_\mu \omega_{\nu,ab} + \omega^c_{\mu,a} \omega_{\nu,cb} - (\mu \leftrightarrow \nu). \tag{2}$$

The gravitational topological charge q_{gr} in fundamental quantum field theory is assumed to be an integer, in Euclidean space,[8] which, generally speaking, cannot be the case in a condensed medium, where the analog of gravitation is associated with the appearance of the order parameter, and total covariance is lacking. We will show that in a ^{3}He-A film the corresponding charge is fractional: $q_{gr} = N/16$. Here N is the internal integer-valued topological invariant of the system, which is defined as the integral of the Green's function over momentum space,[4,7] and which varies discontinuously as one of the energy levels of the transverse motion of the fermions in the film passes through the Fermi level (or more accurately: as the Fermi level passes through the diabolical point in the fermion spectrum in the film).

The orbital part of the order parameter in the ^{3}He-A film is a complex vector (see, e.g., Ref. 7):

$$\vec{\Delta} = \vec{\Delta}_1 + i\vec{\Delta}_2 ,$$

which lies in the x, y plane of the film. In equilibrium $\vec{\Delta}_1 \perp \vec{\Delta}_2$, $|\vec{\Delta}_1| = |\vec{\Delta}_2| = \Delta$, and there is only one degree of freedom—the phase Φ of the condensate

$$\vec{\Delta}^{eq} = \Delta(\hat{x} + i\hat{y}) e^{i\Phi}, \tag{3}$$

We will not restrict the discussion, however, to the equilibrium value. The action for $\vec{\Delta}$ is obtained by integrating over the Bogolyubov fermions ("bogolons"), which in the simple case of independent levels of the transverse motion (the interaction of the fermions with the various levels, as we will see, does not change the result) satisfies the Bogolyubov's equation

$$i \frac{\partial \psi}{\partial t} = [\tau_1 \frac{1}{2} \{\Delta^i_1, p_i\} + \tau_2 \frac{1}{2} \{\Delta^i_2, p_i\} + \tau_3 (\epsilon_n(\mathbf{p} - e\tau_3 \mathbf{A}) - \mu - eA_0)]\psi, \tag{4}$$

where ψ is the Bogolyubov spinor, τ_a are the Pauli matrices in the particle–hole space, $i = (1,2)$, $p_i = -i(\partial/\partial x^i)$, $\{,\}$ is the anticommutator, $\epsilon_n(\mathbf{p}) = \epsilon_n(0) + p^2/2m_n$ is the spectrum excited by the nth level of the transverse motion, where for large n we have $\epsilon_n(0) \sim n^2/a^2$; here a is the film thickness; and A_μ is the gauge field, which is introduced for convenience, and which in an electrically charged system is the electromagnetic field. The spectrum of the Bogolyubov fermions has the form

$$E^2_n(\mathbf{p}) = (\epsilon_n(p) - \mu)^2 + (\vec{\Delta}_1 \mathbf{p})^2 + (\vec{\Delta}_2 \mathbf{p})^2 .$$

In order to find the quantizing charge, we must determine which term in the action varies discontinuously as the nth level passes through the Fermi level $\epsilon_F = \mu$, i.e., as the topological invariant N jumps from $n-1$ to n. Here we assume that the order parameter does not itself change abruptly, since it is induced by other filled levels. To this end, let us consider the result of integrating over the fermions for $\epsilon_n(0)$ near μ on both sides of it. We introduce $M = \epsilon_n(0) - \mu$ and require that $M \ll \Delta^2 m_n$. In this parameter region the characteristic momenta near the minimum of the spectrum $E_n(p)$ are of the order of $p \sim M/\Delta$. We can therefore ignore in $\epsilon_n(p)$ the term with dispersion, since $p^2 m_n \sim (M^2/\Delta^2 m_n) \ll M$. As a result, the Bogolyubov equation transforms into an equation for relativistic fermions with mass M in the triad field

$$\left(\frac{1}{2}\tau^a \left\{e^\mu_a, \frac{\partial}{\partial x^\mu}\right\} + M - eA_0\right)\psi = 0, \tag{5}$$

where $x_0 = it$ and the triads e^μ_a have the form $e^i_1 = \Delta^i_2, e^i_2 = -\Delta^i_1, e^0_0 = 1$. This equation can be rewritten in standard invariant form in terms of the spin connection ω:

$$\tau^c e^\mu_c \left(\partial_\mu - \frac{1}{8}\omega_{\mu,ab}[\tau^a, \tau^b]\right)\psi = -M\psi, \tag{6}$$

where in our case only the component $\omega_{\mu,12} = -\omega_{\mu,21}$ is nonzero, and the temporal component of the connection is determined by the electromagnetic field

$$\frac{1}{2}\omega_{0,12} = eA_0, \tag{7}$$

and the spatial components constitute a generalized superfluid velocity with nonzero curl

$$v_i = \frac{1}{2}\omega_{i,12} = \frac{1}{2}(e_{1i}\partial_k e^k_2 - e_{2i}\partial_k e^k_1), \tag{8}$$

which in the equilibrium vacuum manifold (3) is a potential field: $\mathbf{v}^{eq} = \frac{1}{2}\vec{\nabla}\Phi$. Elevation and elimination of the orbital indices is realized by the metric tensor $g^{\mu\nu} = e^\mu_a e^\nu_a$.

Equation (5) [or, in its rewritten form, Eq. (6)] is invariant with respect to the gauge transformation $U(1)$, upon which

$$\psi \to e^{i\tau_3\alpha}\psi, \quad e^i_1 + ie^i_2 \to e^{2i\alpha}(e^i_1 + ie^i_2), \quad A_0 \to A_0 + \frac{1}{e}\frac{\partial\alpha}{\partial t}, \quad \mathbf{v} \to \mathbf{v} + \vec{\nabla}\alpha. \tag{9}$$

According to Ref. 8, integration over the fermions that satisfy Eq. (6) leads to the desired effective action (1) for the spin connection with the topological charge

$$q_{gr} = \frac{1}{2}\,\frac{1}{16}\,\mathrm{sign}(M), \tag{10}$$

where the additional coefficient 1/2, in comparison with Ref. 8, cancels the doubling of the number of degrees of freedom which comes about as a result of going from particles to Bogolyubov particles and holes. The abrupt change due to a change in the sign of M and, consequently, the passage of one of the levels through the Fermi level, is thus $\Delta q = 1/16$. The corresponding jump in the action, which is expressed in terms of the ^{3}H-A variables, has the form

$$\Delta S = \frac{1}{24\pi}\int d^2x\,dt\,e^{ij}(eA_0\partial_i v_j + v_i\partial_j eA_0 + v_j\partial_t v_i). \tag{11}$$

This expression is an adiabatic invariant of the Chern–Simons type, since it contains the gauge field A_0 in explicit form. It is nevertheless a gauge-invariant expression. The coefficients of such terms cannot depend on the space-time coordinates, and, consequently, cannot vary with adiabatic variation of the system. This is one reason the coefficient of such a term is quantized. Equation (11) can therefore be generalized to the case of interacting levels of the transverse motion, and it is also possible to reconstruct from it the total Chern–Simons action for the case where there are N levels of the transverse motion (allowing for spin) under the Fermi level:

$$S = \frac{N}{24\pi} \int d^2x dt e^{ij} (eA_0 \partial_i v_j + v_i \partial_j eA_0 + v_j \partial_t v_i). \tag{12}$$

In the case of strongly interacting levels, when the concept of the number of levels under the chemical potential loses meaning. Eq. (12) remains valid with this one difference: that N now becomes the topological invariant of the Green's function in momentum space $k_\mu = (\omega, k_x, k_y)$ (Refs. 4 and 7):

$$N = \frac{1}{24\pi^2} e^{\mu\nu\lambda} \int dk_x dk_y d\omega \mathrm{Tr} \{ G \partial_\mu G^{-1} G \partial_\nu G^{-1} G \partial_\lambda G^{-1} \}, \tag{13}$$

This invariant changes abruptly as the chemical potential passes through the diabolical (conic) point in the fermion spectrum. The existence of such an internal invariant of the state is a second reason for quantization of the physical parameter.

The terms in the action which do not contain the gauge field in explicit form or contain it only in gauge-invariant form, for example, in combination with v_μ, i.e., in the form $(eA_\mu - v_\mu)$, do not change abruptly upon passing through the diabolical point. Their coefficients can depend on both space and time. This also applies to the term

$$\int d^2x dt\, \sigma_{xy} e^{ij} (v_0 - eA_0) \partial_i (v_j - eA_j)$$

which was found in Ref. 7, and which describes the internal Hall effect. As was shown in Ref. 7, the Hall conductivity σ_{xy} is quantized only approximately, to the extent of the smallness of Δ/ϵ_F, while the coefficient of the term (12) is quantized exactly.

An interesting consequence of Eq. (11) is the fact that the fermionic charge of a vortex with p circulation quanta, i.e., with $\oint \mathbf{v} d\mathbf{x} = p\pi$, changes by the amount $p/24$ as a result of variation of the topological invariant N.

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