Current-voltage characteristic of extended Josephson junction: analogy with crystal growth kinetics

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An analogy with the kinetics of crystal growth has been pursued to find the shape of the resistance-fluctuation region of the current-voltage characteristic of an extended Josephson junction and also the current-voltage characteristic of a junction in the mixed state with a low vortex density.

Despite the increasing practical applications of the Josephson effect and the nearly 30-year history of research in the field, certain aspects of the effect remain unresolved (see, for example, the monograph by Barone and Paternò[1] and the bibliography there). One example is the current-voltage characteristic of an extended Josephson junction under various external conditions (more on this below). In the present letter we derive the shape of this characteristic on the basis of an analogy with the kinetics of crystal growth.

The Josephson phase φ in the presence of the bias current density j is described by the equation[1]

$$\omega_J^{-2}\ddot{\varphi} + \frac{\hbar}{2eRj_c}\dot{\varphi} = \lambda_J^2\Delta\varphi - \sin\varphi + \frac{j}{j_c}. \tag{1}$$

Here ω_J is the Josephson plasma frequency, R is the total resistance to the normal current per unit area of the extended Josephson junction, j_c is the critical Josephson current density, and λ_J is the Josephson depth to which the magnetic field penetrates. Equation (1) can be derived through a variation of the Josephson Hamiltonian

$$H = \frac{\hbar j_c}{2e}\int d^2x\left[\frac{1}{2}\lambda_J^2(\nabla\varphi)^2 + 1 - \cos\varphi - \frac{j}{j_c}\varphi\right] \tag{2}$$

under the condition that the phase obeys the dynamic equation

$$\frac{\hbar j_c}{2e}\left(\omega_J^{-2}\ddot{\varphi} + \frac{\hbar}{2eRj_c}\dot{\varphi}\right) = -\delta H/\delta\varphi. \tag{3}$$

The latter relations describe the growth of a crystal surface (Ref. 2, for example). The growth dynamics described by (3) is not of a purely relaxation nature. As can be seen from (2), the phase plays the role of the boundary position, and the bias current density plays the role of the supersaturation. At absolute zero, growth does not occur until the supersaturation exceeds a certain threshold. In terms of the extended Josephson junctions, this situation corresponds to the absence of a voltage across the junction at currents below the Josephson current j_c. Above absolute zero, and at a low supersaturation, the crystal grows in a layer-by-layer fashion through the formation and propagation of two-dimensional nucleating regions over its boundary. This regime corresponds to a resistance-fluctuation conductivity of the Josephson junction. We will now show how the ideas of Ref. 2 can be used to find the shape of this part of the current-voltage characteristic.

The energy benefit upon the appearance of a nucleating region—a closed vortex ring of radius r—within the extended Josephson junction occurs because the phase inside this ring is greater by 2π than the phase outside the nucleating region. The energy disadvantage corresponds to the linear energy of the Josephson vortex. The energy of a circular nucleating region is therefore

$$E(r) = 2\pi r\epsilon - \pi r^2 \cdot 2\pi \frac{j}{j_c} \frac{\hbar j_c}{2e},$$

where ϵ is the linear tension of the vortex. A critical ring is characterized by a radius r_c and an energy E_c, given by

$$r_c = \epsilon e/\pi\hbar j \qquad E_c = \epsilon^2 e/\hbar j.$$

A nucleating region of this type is meaningful under the condition $r_c \gg \lambda_J$. It is easy to verify that this inequality is equivalent to the condition that the bias current be small in comparison with the Josephson critical current ($j \ll j_c$). A Josephson junction of area S contains S/r_c^2 possible independent rings, each of which arises with a probability per unit time

$$1/\tau = 1/\tau_0 \exp(-E_c/T),$$

where τ_0^{-1} is a microscopic frequency. In our case we have $\tau_0^{-1} \sim \omega_J$. After a time t, the number of nucleating regions is $(S/r_c^2)(t/\tau)$, and the average distance between them is

$$\delta \sim r_c(\tau/t)^{1/2}.$$

Each ring grows radially at a velocity $v = \mu j$, where μ is the mobility of a vortex. A coalescence of nucleating regions occurs when their radius $r = vt$ reaches the value δ, i.e., under the condition

$$t = \bar{t} = (r_c^2\tau/v^2)^{1/3}, \quad r = \delta = v\bar{t} = (vr_c^2\tau)^{1/3}.$$

The overall average rate of change of the phase is

$$\bar{\dot{\varphi}} = 2\pi/\bar{t} \sim (\hbar^2\mu^2\omega_J/e^2\epsilon^2)^{1/3} j^{4/3} \exp(-\epsilon^2 e/3\hbar jT).$$

The mobility which appears in the last expression is calculated in exactly the same way as the mobility of a step at the surface of a crystal.[2] Equation (1) may be thought of as the equation of motion of a vortex with a corresponding profile of the phase φ $\left[(\tilde{x} = (x - vt/\sqrt{1 - v^2/c_0^2})\right]$ (c_0 is the velocity of Swihart waves[1]):

$$-\frac{\hbar}{2eRj_c}\frac{v}{\sqrt{1-v^2/c_0^2}}\frac{\partial\varphi}{\partial\tilde{x}}=\lambda_J^2\frac{\partial^2\varphi}{\partial\tilde{x}^2}-\sin\varphi+j/j_c.$$

Multiplying both sides by $\partial\varphi/\partial\tilde{x}$, and integrating from $-\infty$ to $+\infty$, we find

$$\frac{\hbar}{2eR}\frac{v}{\sqrt{1-v^2/c_0^2}}\int_{-\infty}^{\infty}\left(\frac{\partial\varphi}{\partial\tilde{x}}\right)^2 d\tilde{x}=2\pi j.$$

In the case $v\ll c_0$ we find the following expression for the mobility:

$$\mu=\pi eR\lambda_J/2\hbar, \tag{4}$$

This expression is meaningful under the condition $(eRj_c/\hbar\omega_J)(j/j_c)\ll 1$. Using the Josephson relation $\overline{\varphi}=(2e/\hbar)\overline{V}$ (V is the voltage) and the latter expression, we find

$$\overline{V}\sim Rj_c\left(\frac{\hbar^2 c_0}{e^2 R\epsilon}\right)^{1/3}(j/j_c)^{4/3}\exp(-\epsilon^2 e/3\hbar jT). \tag{5}$$

Up to this point we have been dealing with the case $j\ll j_c$. In the opposite limit $j_c-j\ll j_c$ we find the following expression, with an exponential accuracy, making use of the results of Ref. 3:

$$\overline{V}\sim\exp\left(-\frac{3\sigma}{4}\frac{\epsilon\lambda_J}{T}\frac{j_c-j}{j_c}\right). \tag{6}$$

Here $\sigma\leqslant 5$ is a numerical constant.

There is yet another example of a current-voltage characteristic which has an analog in crystal growth. Let us consider an extended Josephson junction in a longitudinal magnetic field H at a low temperature. If $H<H_{c1}=4\pi\epsilon/\Phi_0$ (Φ_0 is the magnetic flux quantum), the voltage across the junction at a low current will be determined by the appearance and propagation of vortex rings, discussed above [expression (5)]. At $H\geqslant H_{c1}$, a lattice of Josephson vortices arises within the Josephson junction. This lattice drifts under the influence of the bias current, leading to an increase in the phase and thus to a finite voltage. Each vortex moves at a velocity $v=\mu j$. If the density of vortices is n, the overall rate of increase of the phase is $\dot{\varphi}=nv$. Using (4) and the Josephson relation, we find

$$\overline{V}=\frac{\pi}{4}n\lambda_J Rj. \tag{7}$$

This motion is meaningful if the average distance between vortices exceeds the width of the vortices, i.e., under the condition $n\lambda_J\ll 1$. Current-voltage characteristic (7) corresponds precisely to the growth of a vicinal surface of a crystal, which occurs through a lateral motion of steps.[2] The latter behavior is linear only in the limit $j\to 0$. The reason for the nonlinearity of the current-voltage characteristic is that during the motion of a lattice of vortices at a velocity $v=\mu j$ the lower critical field is itself a function of v: $H_{c1}(v)=H_{c1}/\sqrt{1-v^2/c_0^2}$ (Ref. 1). The nonlinearity effects are seen most obviously in fields near the lower critical field H_{c1}, where the density of vortices

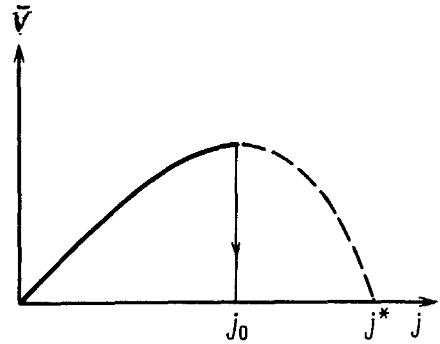

FIG. 1.

is logarithmically small (as in the situation in which a system is near the point of a transition from a commensurable phase to an incommensurable phase[4]). As a result, in a fixed external field the voltage should vanish at $j = j^* = (2\sqrt{2}\hbar\omega_j/\pi eR)(H - H_{c1}/H_{c1})^{1/2}$ in accordance with

$$\bar{V} \sim Rj^*/\ln(j^*/(j^*-j)) .$$

The initial part of the current-voltage characteristic is described by $(j \to 0)$ Consequently, the current-voltage characteristic of a junction in a magnetic field near H_{c1} (Fig. 1) has a maximum $j_0 < j^*$ and also an unstable region (the dashed region), which corresponds to a negative differential resistance. Just where this instability will lead is difficult to say. It seems reasonable to suggest that when the maximum of the current-voltage characteristic is reached the voltage will abruptly vanish (the arrow in Fig. 1), and the vortices will disappear from the Josephson junction (there will essentially be a transition to the resistance-fluctuation part of the current-voltage characteristic discussed above).

Curiously, the dependence of the crystal growth velocity on the supersaturation which was found in Ref. 5 is also an analog of the current-voltage characteristic which was established a fairly long time ago by Aslamazov and Larkin[6] for the case $j \gg j_c$.

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$$\bar{V} \sim Rj/\ln(H_{c1}/(H-H_{c1})).$$

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