Generating correlation function in 2d conformal theory and the Sugawara construction of the 2+1 Chern–Simons theory

Ya. I. Kogan and V. V. Fock
Institute of Theoretical and Experimental Physics, Academy of Sciences of the USSR

(Submitted 24 January 1990)
Pis'ma Zh. Eksp. Teor. Fiz. **51**, No. 4, 184–187 (25 February 1990)

A generating correlation function of the currents in the *WZW* model is constructed on the basis of the invariance of the vacuum wave function of the 2 + 1 Chern–Simons theory under the diffeomorphisms and gauge transformations.

1. Witten[1] and Moore and Seiberg[2] have discovered that there is a relationship between the *WZW* 2*d* conformal model and the 2 + 1 Chern–Simon theory, which casts the conformal theories in an entirely different light. We will construct a generating function for chiral currents $J(z)$ and for the energy-momentum tensor $T(z)$ in the WZW model, expressed as a vacuum-state wave function of the CS theory. This procedure immediately gives rise to the Sugawara construction: $T(z) = (1/k + c_v):J(z)J(z):$ which is necessary for the study of the conformal properties of *WZW*.[3]

Let G denote the real semiordinary Lie group. $\mathcal{G}$ the corresponding Lie algebra, and M the 3D manifold with an edge ∂M. Let us consider the Chern–Simons action

$$S = \frac{k}{4\pi}\mathrm{Tr}\int_M (AdA + \frac{2}{3}A^3) = \frac{k}{4\pi}\mathrm{Tr}\int_M d^3x\, \epsilon^{\mu\nu\lambda}(A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda), \quad (1)$$

where $A_\mu = A^a_\mu T^a$, and $\mathrm{Tr}T^a T^b = 1/2\partial^{ab}$. The action (1) is, within the limiting terms, invariant under gauge transformations

$$A \to g^{-1}Ag + ig^{-1}dg \qquad \text{or} \qquad A_\mu \to g^{-1}A_\mu g + ig^{-1}\partial_\mu g\,,$$

where g is the function on M with the values in G. It is also invariant under the diffeomorphisms f of the manifold M.

Infinitesimally, these transformations are

$$\begin{aligned} g:&\qquad A \to A + d\epsilon - i\,[A, \epsilon] \\ f:&\qquad A \to A + L_v A, \end{aligned} \qquad (2)$$

or in the components

$$\begin{aligned} &A^a_\mu \to A^a_\mu + \partial_\mu\epsilon^a + f^{abc}A^b_\mu\epsilon^c \\ &A^a_\mu \to A^a_\mu + v^\nu\partial_\nu A^a_\mu + A^a_\nu\partial_\mu v^\nu, \end{aligned} \qquad (3)$$

where $\epsilon = \epsilon^a T^a$ is a function with the values in $\mathcal{G}$, and $v = v^\mu\partial_\mu$ is a vector field on M which is tangent to the edge ∂M. According to the Noether theorem, these transformations correspond to the remaining quantities–the generators of the corresponding transformations

$$H_\epsilon = -\frac{ik}{4\pi}\int_{\partial M} d^2x\,\epsilon^a \epsilon^{ij} F^a_{ij}\,; \qquad \epsilon^{ij} F^a_{ij} = \epsilon^{ij}(\partial_i A^a_j + f^{abc} A^b_i A^c_j),$$

$$H_v = \frac{ik}{2\pi}\int_{\partial M} v^k A^a_k \epsilon^{ij}\partial_i A^a_j\, d^2x\,. \tag{4}$$

2. Let us quantize the Chern–Simons theory, making use of the holomorphic representation[4–6] which is equivalent to the canonical quantization in the Schrödinger representation of the low-energy limit of the topologically massive theory.[7,8] We chose on ∂ M a complex structure τ and a corresponding complex coordinate z, which allows us to represent the 1-form $A_i dx^i$ on ∂M as a sum (1,0) of the form Adz and (0,1) of the form Adz. The Hilbert space $\mathscr{H}$ is the space of the analytic functions (functionals) $F(A)$ on the form space of the type (1,0). The scalar product on $\mathscr{H}$ is given by a path integral

$$\langle F(A), G(A)\rangle = \int DAD\bar{A}\, e^{-\frac{k}{\pi}\int_{\partial M} A\bar{A}}\; \overline{F(A)}\, G(A\;). \tag{5}$$

This product is determined by the commutator $[A(x),\bar{A}(y)] = (\pi/k)\delta(x-y)$, which follows from (1). The function $\bar{A}(x)$ corresponds to the operator $(\pi/k)[\delta/\delta A(x)]$. The quantum analogs (4) are

$$H_\epsilon = -\frac{ik}{\pi}\,\mathrm{tr}\int_{\partial M}(\bar{\partial}\epsilon A - \frac{i\pi}{k}\partial\epsilon\frac{\delta}{\delta A} + \frac{i\pi}{k}[\epsilon, A\;]\frac{\delta}{\delta A}),$$

$$H_v = \frac{ik}{\pi}\,\mathrm{tr}\int_{\partial M}(\bar{\partial}A - \frac{2\pi}{k}\partial\frac{\delta}{\delta A})(Av + \frac{2\pi}{k}\bar{v}\frac{\delta}{\delta A}). \tag{6}$$

3. Let us consider in $\mathscr{H}$ space the set of vectors

$$\chi_{\mu,\xi}(A) = \exp\{\frac{ik}{2\pi}\,\mathrm{Tr}\int_{\partial M}\mu A^2 + k\,\mathrm{Tr}\int_{\partial M}\xi A\} \tag{7}$$

which depend on the functional parameters μ and ξ, where $\xi = \xi^a T^a$ is a $\mathscr{G}$-valued (0,1) differential, and μ is a $(-1,1)$ differential (the Beltrami differential) which satisfies the condition $|\mu| < 1$. Note that such a Beltrami differential determines on ∂M a different complex structure, specifically, that in which the 1-form $dz + \mu d\bar{z}$ is of the type (1,0). The condition $|\mu| < 1$ guarantees that this form is not real.

Let $\mathscr{H}'$ denote the space of the analytic functionals from μ and ξ. Let us determine the $\mathscr{H}\to\mathscr{H}'$ with the help of a scalar product. $F(A)\to\Phi_F(\mu,\xi) = \langle F(A),\chi_{\mu,\xi}(A)\rangle$. A transition from $\mathscr{H}$ to $\mathscr{H}'$ may be considered as a replacement of the basis in $\mathscr{H}$. The action of the operators on $\mathscr{H}$ carries over to the action on $\mathscr{H}'$

in the following way:

$$\hat{X}\,\Phi_F(\mu,\xi) = \Phi_{XF}(\mu,\xi)^{\cdot} = \langle\, F(A),\ X^{*}\chi_{\mu,\xi}(A)\rangle\ ,$$

where X is the operator on $\mathscr{H}$. It is easy to verify that operators (6) in the μ,ξ representation have the form

$$\hat{H}_\epsilon = \mathrm{Tr}\int_{\partial M}(\bar\partial - \mu\partial - ad_\xi)\,\epsilon\,\frac{\delta}{\delta\xi} - k/\pi\,\mathrm{Tr}\int_{\partial M}\xi\,\partial\epsilon\ ,$$

$$\hat{H}_v = \int_{\partial M}(\bar\partial - \mu\partial + \partial\mu)(v + \mu\bar v)\,\frac{\delta}{\delta\mu} + \mathrm{Tr}\int_{\partial M}\Big[(v\partial + \bar v\bar\partial + (\bar\partial - \mu\partial)\bar v)\xi\,\frac{\delta}{\delta\xi} + \frac{k}{\pi}\bar v\,\xi\,\partial\xi\Big] - \frac{c}{12\pi}\int_{\partial M}(v + \mu\bar v)\,\partial^2\mu\ . \tag{8}$$

It is necessary in this case to impose a regularization which does not disrupt the natural commutation relations

$$[H_{v_1}, H_{v_2}] = H_{[v_1,v_2]}\,,\qquad [H_{\epsilon_1}, H_{\epsilon_2}] = H_{[\epsilon_1,\epsilon_2]}\,,\qquad [H_v, H_\epsilon] = H_{L_v\epsilon}\,.$$

The constant c can be arbitrary. In particular $c = 0$ when the regularization involves a separation of points.

It is obvious that the vacuum state of the theory must be invariant under the symmetry transformation (2) and, hence, its wave function Φ_0 in μ,ξ representation must satisfy the equations

$$\hat{H}_\epsilon\,\Phi_0(\mu,\xi) = \hat{H}_v\,\Phi_0(\mu,\xi) = 0. \tag{9}$$

Using the explicit form of the operators $\hat{H}_\epsilon$ and $\hat{H}_v$ in μ,ξ representation (8), we can show[9] that the function Φ_0 is a generating function of the chiral-current correlators and of the energy-momentum tensor in the conformal *WZW* theory, with $(k\,\dim G/k + c_v)$. In other words,

$$(4\pi i)^m\,\frac{1}{\Phi_0}\,\frac{\delta}{\delta\xi^{a_1}(z_1)}\cdots\frac{\delta}{\delta\xi^{a_n}(z_n)}\,\frac{\delta}{\delta\mu(w_1)}\cdots\frac{\delta}{\delta\mu(w_m)}\Big|_{\xi=0} = \langle J^{a_1}(z_1)\ldots J^{a_n}(z_n)\,T(w_1)\ldots T(w_m)\rangle\ , \tag{10}$$

where the correlation function on the right side corresponds to a complex structure given by μ. This expression is written for ∂M of type O, without specifying the points, when Φ_0 is uniquely determined by Eq. (9). In the case of surfaces with the specified points and/or higher order, there arises a finite-dimensional space of the vacuum wave functions, which is isomorphic to the space of conformal blocks of the corresponding conformal theory. The explicit isomorphism is given by expression (10).

To prove (10), we need only to verify the relations

$$\pi^2 \frac{\delta}{\delta \xi^a(z)} \frac{\delta}{\delta \xi^b(w)} \Phi_0 = \Phi_0 \frac{k\delta^{ab}\partial u(z)\partial u(w)}{(u(z)-u(w))^2} + \frac{\pi f^{abc}\partial u(z)}{u(z)-u(w)} \frac{\delta \Phi_0}{\delta \xi^c(z)} + O(1) \tag{11a}$$

$$\pi^2 : \frac{\delta}{\delta \xi^2} : \Phi_0 = \pi^2 \lim_{z \to w} \operatorname{tr} \frac{\delta}{\delta \xi(z)} \frac{\delta \Phi_0}{\delta \xi(w)} \cdot - \frac{k \dim G}{(u(z)-u(w))^2} \partial u(z)\partial u(w) \Phi_0 \tag{11b}$$

$$+ k \frac{\dim G}{6} s(u)\Phi_0) = (2\pi)\,(k+c_v)\frac{\delta \Phi_0}{\delta \mu(z)},$$

$$[: \frac{\delta^2}{\delta \xi^2} : \frac{\delta}{\delta \mu}] = 0, \tag{11c}$$

where c_v is the dual number of the Coxter algebra $\mathscr{G}$, $\mu = \bar{\partial}\mu/\partial\mu$, and the Schwarz derivative is $s(\mu) = \partial^3\mu/\partial\mu - 3/2(\partial^2\mu/\partial\mu)^2$. Relations (11) correspond to an operator expansion of the Kac-Moody chiral currents:

$$J^a(u_1)J^b(u_2) = \frac{k\delta^{ab}}{(u_1-u_2)^2} + \frac{f^{abc}J^c(u_1)}{u_1-u_2} + O(1)$$

and the Sugaware construction

$$T(u) = \frac{1}{k+c_V} :J^a(u)J^a(u).$$

Here ${}^{\delta}/_{\delta\xi^a(z)} \to J^a(z)$, and ${}^{\delta}/_{\delta\mu(z)} \to T(z)$.

We note that the relation $\delta^2\Phi_0/\delta\xi\delta\xi = .k(\delta\Phi_0/\delta\mu)$ follows from (7). The renormalization $k \to k + c_V$ occurs as a result of the regularization of H_v in the transition from a second-order operator in A representation (6) to a first-order operator in $\mu\xi$ representation (8). This renormalization can be regarded as a certain normal ordering.

We wish to thank A. S. Losev for useful discussions.

After this letter was submitted to publication, we received a preprint from H. Verlinde, PUPT-89/1140, in which he analyzed equations such as (8b), which were derived from the conformal Ward identities. These equations admittedly describe only the chiral diffeomorphisms.

[1]E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[2]G. Moore and N. Seiberg, Phys. Lett. B**220**, 422 (1989).
[3]V. G. Knizhnik and A. B. Zamalodchikov, Nucl. Phys. B**241**, 33 (1984).
[4]M. Bos and V. Nair, Phys. Lett. B **223**, 61 (1989).
[5]S. Elitzur *et al.*, Preprint IASSNS-HEP-89-20.
[6]J. M. F. Lavastida and A. V. Ramallo, Phys. Lett. B**228**, 214 (1989).
[7]G. V. Dunne *et al.*, Ann. Phys. **194**, 197 (1989).
[8]Ya. I. Kogan, Preprint ITEP 163-89.
[9]V. V. Fock and Ya. I. Kogan, Preprint ITEP, to be published.

Translated by S. J. Amoretty