Canonical quantization of relativistic particle

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In a new interpretation, a reparametrization-invariant action of a relativistic particle—at the classical level—describes a particle and an antiparticle simultaneously. A canonical gauge in which a description of this sort is realized in a natural way is pointed out. A canonical quantization of a spinor particle in such a gauge is carried out.

The theory of a relativistic particle is currently attracting interest in connection with the string quantization problem.[1] No systematic procedure for a canonical quantization for this theory has so far been presented.[2–5] In the present letter we offer a new interpretation of the corresponding classical theory which makes possible a systematic canonical quantization.

As a preliminary step we consider the reparametrization-invariant action of a scalar particle:

$$S=\int L d\tau, \quad L=-m\sqrt{\dot{x}^2}, \quad \dot{x}^\mu = dx^\mu/d\tau, \quad \eta_{\mu\nu}=(1,-1,\ldots). \tag{1}$$

The relations $\pi_\mu = \partial L/\partial \dot{x}^\mu$ here make it possible to express the three velocities $\dot{x}^i$ and also the sign of the velocity $\dot{x}_0^i$ in terms of the momenta and $|\dot{x}_0|$: $\dot{x}^i = |\dot{x}_0|\pi_i(\pi_k^2+m^2)^{-1/2}$, $\operatorname{sign}\dot{x}_0 = \zeta = -\operatorname{sign}\pi_0$. In addition, these relations lead to the constraint $\pi^2 = m^2$, which is a unique constraint here (of type 1). The Hamiltonian $H^{(1)}$, which is constructed by the canonical procedure and which determines the evolution, is $H^{(1)} = \lambda\left[\sqrt{\pi_k^2+m^2} - |\pi_0|\right]$, $\lambda = |\dot{x}_0| > 0$. This is a gauge theory. A gauge is imposed in order to eliminate the arbitrariness (in λ). We choose the gauge

$$x^0 = \zeta\tau, \tag{2}$$

which yields $\lambda = 1$. The variable $\zeta = -\operatorname{sign}\pi_0$ is not fixed by the constraints and the gauge condition. We will interpret trajectories with $\zeta = 1$ as the trajectories of a particle, while trajectories with $\zeta = -1$ are the trajectories of an antiparticle. The imposition of an external field supports this interpretation, since trajectories with a given ξ correspond to a particle with a charge ζe. If there is no electromagnetic field, two points of view are possible. First, we could simply identify these two classes with each other, or discard one of them, imposing yet another additional condition, e.g., $\zeta = 1$. That approach would be equivalent to replacing (2) by the gauge $x^0 = \tau$, as is usually done. From the standpoint of a Lagrangian formalism, this means that the group of gauge transformations of action (1) contains transformations $x^\mu(\tau) \to x^\mu(-\tau)$, in addition to transformations $x^\mu(\tau) \to x^\mu(f(\tau))$, $\dot{f} > 0$. In this case, gauge (2) does not break the symmetry associated with a change in the sign of τ. The condition $x^0 = \tau$ does break this symmetry. One could say that action (1) with this expanded symmetry group describes only one neutral particle. The other point of view is that the gauge group contains only transformations $x^\mu(\tau)\to x^\mu(f(\tau)), f>0$. In this case, trajectories with different ζ should be regarded as distinct. Action (1) with a gauge group of this sort describes both a particle and an antiparticle. When there is an external field, the transformation $x^\mu(\tau)\to x^\mu(-\tau)$ is not a symmetry transformation of the action. This means that if a gauge group which includes such a transformation is retained it will not be possible to introduce a minimal electromagnetic interaction, in agreement with the interpretation offered above for the description of a neutral particle. It is thus found—at the classical level—that the possibility of a simultaneous description of both a particle and an antiparticle is actually built into action (1). The gauge proposed in (2) makes it possible to actually realize this possibility.

We carry out the quantization immediately for a spinor particle in a canonical fashion, in gauge (2). The Lagrangian of the spinor particle is[2]

$$L=-\frac{\dot{x}^2}{2e}-\frac{e}{2}m^2-i(\psi\dot{\psi}-\psi_5\dot{\psi}_5)+i\chi\left(\frac{\dot{x}\psi}{e}+m\psi_5\right),$$

where x^μ and e are even variables; ψ^μ, χ, and ψ_5 are odd variables; and $\mu=0, 1, \ldots D-1$. In the Hamiltonian formalism we have the constraints

$$P_e=\Pi_\mu+i\psi_\mu=P_\chi=P_5-i\psi_5=\sqrt{\pi_k^2+m^2}-|\pi_0|=\pi\psi-m\psi_5=0, \qquad (3)$$

where π_μ, P_e, Π_μ, P_χ, and P_5 are the momenta which are the conjugates of x^μ, e, ψ^μ, χ, and ψ_5, respectively. Among constraints (3) we can distinguish four independent type 1 constraints, so by switching to a canonical gauge we find the four auxiliary conditions $x^0-\zeta\tau=\chi=P_5=e-|\pi_0|^{-1}=0$, where $\zeta=-\operatorname{sign}\pi_0$. Replacing x^0 by the variable $x^{0\prime}=x^0-\zeta\tau$, we make a transformation in a canonical way to constraints which do not depend on the time. We can thus make direct use of the "Dirac" quantization method for theories with type 2 constraints. After this canonical transformation, the Hamiltonian on the constraints surface is $H=\sqrt{\pi_k^2+m^2}$. Eliminating the variables e, P_e, χ, P_χ, ψ_5, P_5, $x^{0\prime}$, $|\pi_0|$ by means of the constraints, we verify that for all remaining variables x^k, π_k, ζ, ψ^μ, Π_μ the Dirac brackets in terms of all constraints reduce to Dirac brackets in terms of the remaining constraints, $\Pi+i\psi=\pi\psi=0$. The algebra of commutation relations for the operators of the corresponding quantum mechanism is determined by these Dirac brackets. We construct a realization of this algebra for $D=4$ in the Hilbert space of four-component columns f. It is

$$\hat{\zeta}=\gamma^0,\quad \hat{\pi}_k=\hat{p}_k I,\quad \hat{x}^k=x^k I+\frac{\epsilon^{kij}\Sigma^i\hat{p}^j}{2m(\hat{\omega}+m)},$$

$$\hat{\psi}^0=\frac{1}{2m}\vec{\Sigma}\hat{\mathbf{p}},\quad \hat{\psi}^k=\gamma^0\left(\frac{1}{2}\Sigma^k+\frac{\hat{p}^k\vec{\Sigma}\hat{\mathbf{p}}}{2m(\hat{\omega}+m)}\right),$$

where $\hat{p}^k=-i\partial_k$, $\hat{\omega}=\sqrt{\hat{p}_k^2+m^2}$, and I is the 4×4 unit matrix. The time evolution of the state vectors is determined by the Hamiltonian which we have found in accordance with the Schrödinger equation. Written in terms of the "physical" time $x^0 = \zeta\tau$, where ζ are the eigenvalues of the operator $\hat{\zeta}$, the Schrödinger equation is

$$i\partial f/\partial x^0 = \gamma^0 \hat{\omega} f. \tag{4}$$

Equation (4) is the Dirac equation in the Foldy–Wouthuysen representation.[8] Making the unitary transformation $f = U\Phi$, where $U = (\hat{\omega} + m + \gamma\hat{p})/(2\hat{\omega}(\hat{\omega} + m))^{1/2}$, we find the Dirac equation in the standard representation for ψ: $(\gamma^\mu \hat{p}_\mu - m)\psi = 0$, $\hat{p}_\mu = i\partial_\mu$. Applying this transformation to the operators $\hat{x}^k$ and $\hat{\psi}^\mu$ we find

$$U^+ \hat{x}^k U = \hat{X}^k = x^k I + \frac{i}{2m}\left(\gamma^k - \hat{p}^k \frac{\vec{\gamma}\vec{p}}{\hat{\omega}^2}\right),$$

$$U^+ \hat{\psi}^\mu U = \hat{\Psi}^\mu = \frac{i}{2m}\gamma^5 \sigma^{\mu\nu}\hat{p}_\nu, \quad \hat{p}_0 = \vec{\alpha}\vec{p} + \beta m.$$

The operator $\hat{\Psi}^\mu$ on solutions of the Dirac equation is proportional to the Pauli–Lyuban'skiĭ spin vector, and $\hat{x}^k$ and $\hat{X}^k$ are operators of intermediate position, which were derived some time ago by Pryce[9] from considerations regarding the covariance of expectation values.

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