Regularized scalar Green's function on Riemann surfaces

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The general structure of the scalar Green's function on the compact Riemann surfaces has been studied. An equation which couples the second- (and higher-order) derivatives of the scalar propagator with the first derivative has been derived.

The scalar two-point functions on compact Riemann surfaces play an important role in string perturbation theory. Various mathematical aspects of these entities were analyzed by many authors and comprehensively discussed in a review by D'Hoker and Phong.[1] At the same time, the two-point Green's function is a scalar propagator in the quantum field theory on a Riemann surface. We will show that some interesting information on the structure of this entity can be obtained by analyzing the auxiliary two-dimensional quantum theory.

Let us consider the theory of scalar field $X(z)$ on Riemann surface M of type h with the metric $ds^2 = 2g_{z\bar{z}}dzd\bar{z}$

$$S = \frac{1}{4\pi}\int d^2z\,[\partial_z X\,\partial_{\bar{z}} X - \Lambda_{\bar{z}}^{z}(\partial_z X)^2 - \Lambda_{z}^{\bar{z}}(\partial_{\bar{z}} X)^2 + 4\pi g_{z\bar{z}}\Phi X], \tag{1}$$

where $\Lambda_{\bar{z}}^{z}(z)$ and $\Phi(z)$ are the background 2D fields, and $\Lambda_{\bar{z}}^{z}$ is an infinitesimal field. Our notation is the same as that used in Ref. 1.

Let us calculate the Λ–Φ component of the effective action (EA) $W(\Lambda,\Phi)$ which is defined as

$$\exp(-W) = \int DX \exp(-S).$$

In perturbation theory this can be accomplished in two ways:

a) By choosing S_0 as the free action

$$S_0 = \frac{1}{4\pi}\int d^2z\,\partial_z X\,\partial_{\bar{z}} X.$$

The Λ–Φ contribution will then be described by diagram *a* shown in Fig. 1.

b) By defining the other free action S_0

$$S_0 = \frac{1}{4\pi}\int d^2z[\partial_z X\partial_{\bar{z}} X - \Lambda_{\bar{z}}^{z}(\partial_z X)^2 - \Lambda_{z}^{\bar{z}}(\partial_{\bar{z}} X)^2],$$

which can be rewritten in covariant form using covariant derivatives $D_z = \hat{\nabla}_z - \Lambda_z^{\bar{z}}\hat{\nabla}_{\bar{z}} + (\hat{\nabla}_{\bar{z}}\Lambda_z^{\bar{z}})\hat{M}$. In this case all Λ–Φ terms are found in diagram *b* in Fig. 1, where the Green's function $G'(z,z)$ is constructed from the new metric

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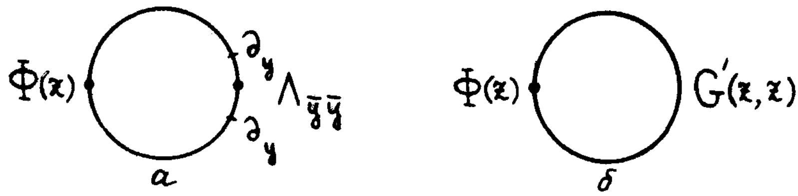

FIG. 1.

$$ds'^2 = 2g_{z\bar z}|\,dz + \Lambda^{z}_{\bar z}|d\bar z|^2. \tag{2}$$

It is evident from expression (2) that the dependence $G'(z,z)$ on Λ is determined by the transformation properties of the function $G(z,z)$ relative to the infinitesimal Beltrami deformations. The general structure of the scalar propagator must therefore be known in the range of the coincident points. Note that the law governing the transformation of the (regularized) scalar Green's function for the Weyl extension of the metric was determined in Ref. 3, but its structure has not been studied completely.

The Green's function is poorly determined in the limit $z\to z'$. To give $G(z,z')|_{z\to z'}$ a meaning, we will regularize the Green's function, making use of the intrinsic time cutoff.[4] It can be shown that in this regularization the following equality holds:

$$\lim_{z\to z'} G(z,z';\epsilon) \equiv G(z;\epsilon) = A + \frac{1}{4\pi}\int d^2\omega\sqrt{g}\,G(z,\omega;\epsilon)N(\omega;\epsilon),$$

$$A \equiv \frac{1}{\int d^2y\sqrt{g}}\int d^2\omega\sqrt{g}\,G(\omega;\epsilon) = -\ln\epsilon + \ldots, N(z;\epsilon) \equiv \Delta_0 G(z;\epsilon) = 2(\Delta_0 - \nabla_z\nabla^\omega$$

$$-\nabla^z\nabla_\omega)\,G(z,\omega;\epsilon)|_{z=\omega}. \tag{3}$$

Here ϵ is an infinitesimal regularization parameter, Δ_0 is a Laplacian, and the Green's function $G(z,\omega)$ satisfies the equation

$$\Delta_0 G(z,\omega) = \frac{4\pi}{\sqrt{g}}\delta(z-\omega) - \frac{4\pi}{\int d^2y\sqrt{g}}. \tag{4}$$

Using the integral representation for the propagator $G(z,\omega;\epsilon)$, we easily find

$$N(z;\epsilon\to 0) = 2\tilde{R}_g(z) \equiv 2R_g(z) + 4\pi g^{z\bar z}\sum_{I,J=1}^{h}\omega^I_z(\mathrm{Im}\,\Omega)^{-1}_{IJ}\bar\omega^J_{\bar z}, \tag{5}$$

where $R_g(z)$ is a scalar curvature, ω^I_z are holomorphic differentials, and Ω_{IJ} is the matrix of the periods of the Riemann surface of type h. It is interesting to note that the integral $\int d^2z\sqrt{g}\tilde{R}_g = 4\pi$ does not depend on the type of surface.

From the equation

$$\delta_{Weyl}G(z,y) = \frac{-2}{\int d^2v\sqrt{g}}\int d^2\omega\sqrt{g}\,\sigma(w)[G(z,w) + G(y,w)]$$

we find the constant A

$$A=-\ln\epsilon+\frac{1}{2\pi}\int d^2y\sqrt{\hat{g}}[\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\rho\partial_\nu\rho+\rho\tilde{R}_{\hat{g}}]$$

$$-\frac{1}{16\pi^2}\int d^2\omega\sqrt{\hat{g}}\int d^2y\sqrt{\hat{g}}\tilde{R}_{\hat{g}}(\omega)G(\omega,y)\tilde{R}_{\hat{g}}(y)+\Psi(m_i). \qquad (6)$$

Here $\hat{g}_{\mu\nu}=e^{-2\rho}g_{\mu\nu}$ is a constant curvature metric, and $\Psi(m_i)$ is a function on the Teichmüller space. Expressions (5) and (6) completely determine the structure of the regularized Green's function (3).

The dependence of the function $G(z;\epsilon)$ on the Beltrami differentials Λ now can be determined by using Eqs. (3), (5), and (6), and also the transformation property

$$\delta_{Beltrami}G(z,y)=\frac{1}{2\pi}\int d^2\omega\Lambda_{\bar{\omega}}^{\omega}\partial_\omega G(z,\omega)\partial_\omega G(y,\omega)+\text{h.c.} \qquad (7)$$

The foregoing results can be used to determine the relevant terms in the effective action. From the requirement that the Λ–Φ terms, found by methods a) and b), be equal, we find the equation for the scalar Green's function

$$(\partial_z G(z,\omega))^2=\nabla_z^2 G(z,\omega)+\frac{1}{2\pi}\partial_z G(z,\omega)\int d^2y\sqrt{g}\nabla_z G(z,y)\tilde{R}_g(y)$$

$$-2\int d^2y\,\partial_z G(z,y)\,\partial_y G(\omega,y)\mathscr{P}_{z\bar{y}}$$

$$+\Psi_{zz}+\frac{1}{8\pi^2\int d^2u\sqrt{g}}\int d^2v\sqrt{g}\nabla_z G_z^{(+)}(z,v)^v\int d^2y\sqrt{g}\,\partial_v G(v,y)\tilde{R}(y). \qquad (8)$$

Here $\mathscr{P}_{z\bar{y}}$ is a projection operator on the space of the holomorphic differentials

$$\mathscr{P}_{z\bar{y}}\equiv\frac{1}{2}\sum_{I,J=1}^{h}\omega_z^I(\operatorname{Im}\Omega)^{-1}_{IJ}\bar{\omega}_{\bar{y}}^J;$$

Ψ_{zz} is a quadratic holomorphic differential, and $\partial_{\bar{z}}\Psi_{zz}=0$; the Green's function $G_z^{(+)}(z,v)^v$ satisfies the equation

$$\Delta_1^{(+)}G_z^{(+)}(z,v)^v=4\pi\delta(z,v).$$

Equation (8) holds when $h\geqslant 2$, since there are no null modes of the Laplacian $\Delta_1^{(+)}$ in this case.[1] The cases in which $h=0,1$ are analyzed below.

Relation (8) which we found is important in that it links the second (and higher-order) derivatives of the Green's function with the first derivatives. Remarkably, Eq. (8) is covariant under the Weyl transformation, which can be proved by directly varying diagrams a) and b) in Fig. 1.

Let us analyze Eq. (8) for the Riemannian surfaces of type $h=0,1$. A sphere

($h=0$) has no Abelian or quadratic differentials. As a result, relation (8) can be written in simpler form

$$(\partial_z G(z,\omega))^2 = \nabla_z^2 G(z,\omega), \quad h=0. \tag{9}$$

It is easy to verify that the Green's function

$$G(z,z') = -\ln \frac{|z-z'|^2}{(1+|z|^2)(1+|z'|^2)},$$

which corresponds to the standard metric on the sphere,[1] satisfies Eq. (9).

In the case of a torus ($h=1$) $\omega_z = 1$, and $\Psi_{zz} = \text{const}$. Equation (8) can therefore be written in the form[1)]

$$(\partial_z G(z-\omega))^2 = \partial_z^2 G(z-\omega) + \frac{1}{\tau_2}\int d^2y\, \partial_z G(z-y)\partial_\omega G(\omega-y) + \text{const}, \quad h=1. \tag{10}$$

The last term in (8) vanishes because of the relation $\int d^2 y G(z-y) = 0$ (Refs. 1 and 3).

The scalar Green's function on the torus is constructed in the terms of θ_1: the Riemann function[1)]

$$G(z-z') \doteq -\ln \left| \frac{\theta_1(z-z';\tau)}{\theta_1'(0,\tau)} \right|^2 - \frac{\pi}{2\tau_2}(z-\bar{z}-z'+\bar{z}')^2. \tag{11}$$

To prove Eq. (10), we note that the Weierstrass ρ-function $\rho(z-\omega) \equiv \partial_z^2 G(z-\omega)$ is holomorphic when $z\neq\omega$ and that it has a Laurent expansion

$$\rho(z-\omega) = \frac{1}{(z-\omega)^2} + \text{regular terms.}$$

On the other hand, it is easy to show that the combination

$$(\partial_z G(z-\omega))^2 - \frac{1}{\tau_2}\int d^2y\, \partial_z G(z-y)\partial_\omega G(\omega-y) \tag{12}$$

is also holomorphic when $z\neq\omega$ and that it has the same pole as ρ. This means that the functions $\rho(z-\omega)$ and (12) differ only by a constant, incomplete agreement with Eq. (10).

Note that identity (10) gives a new integrated representation of the Weierstrass function only in terms of the function θ_1. Furthermore, it follows from relations (3), (5), and (6) that $G(z,z;\epsilon)$ is a constant for the sphere and torus.

The principal result obtained by us is Eq. (8), which is based on the condition that quantum field theory (1) be consistent on the Riemann surface. Accordingly, we have demonstrated how some nontrivial mathematical information can be extracted from the physical theory.

We believe that these results could be useful in calculating higher-order loop corrections in the string equations of motion on the basis of the σ-model method.[5]

[1)] We used the standard parametrization of the torus.[1]

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