Nonintegrability and stationary solitons of complex profile for Landau-Lifshitz equation

L. M. Lerman
Scientific-Research Institute of Applied Mathematics and Cybernetics, Gorki State University

(Submitted 31 January 1990)
Pis'ma Zh. Eksp. Teor. Fiz. **51**, No. 6, 336–338 (25 March 1990)

The Landau-Lifshitz equations with a fourth-order anisotropy energy are shown to be nonintegrable. It is also shown that among the solutions of these equations there are some stationary solitons of complex profile which can be described by symbolic dynamics.

The Landau-Lifshitz (LL) equation $m_t = [m, \delta F/\delta m]$, which is one of the basic equations in the theory of magnetic media, is specified by the density of the free-energy functional $F(m, \nabla m)$, where m is the unit 3-vector of the magnetic moment, and $\delta F/\delta m$ is a variational derivative of the free energy. For a uniaxial ferromagnet, one of the simplest forms of the function F is[1]

$$F = \frac{1}{2}\left(\frac{\partial m}{\partial x}\right)^2 - \frac{1}{2} m \cdot J_e m - \frac{\nu}{4}(m \cdot J_0 m)^2 , \tag{1}$$

where $J_\epsilon = -\mathrm{diag}(1+\epsilon,1,0), \epsilon>0$. At $\nu=0$, the LL equation is integrable, and its exact solutions, including soliton solutions, can be found by a variety of methods. At $\nu\neq 0$, there are no general methods for deriving exact solutions, so it is important to find certain classes of exact solutions, e.g., stationary or traveling waves. We will prove that soliton solutions exist at $\nu\neq 0$, and we will prove that the equation is not integrable for stationary waves. Let us consider the form of stationary wves for an LL equation with F as in (1). Using the notation $\xi = x - ut$ and $' = d/d\xi$, and introducing the new variables $M = um + [m,m']$, we find

$$M' + [m, J_e m] - \nu(m \cdot J_0 m)[m, J_0 m], \qquad m' = [M, m],$$
$$m^2 = 1, \qquad M \cdot m = u. \tag{2}$$

System (2) is a system of Hamilton's equations with two degrees of freedom. The phase manifold N is a cotangent stratification on the 2D sphere T^*S^2; the Poisson brackets are given in Ref. 2. The Hamiltonian of the system is a restriction on the function $H = (M^2 + m\cdot J_e m)/2 - \nu(m\cdot J_0 m)^2/4$ on N. With $\nu=0$, system (2) is integrable, and the Clebsch integral $Q = (M\cdot J_e M - J_e^{(1)} J_e^{(2)} m_3^2)/2$ serves as an additional integral. For all u and for sufficiently small values of $|\nu|$, there exist three pairs of singular points which correspond to stationary solutions of the LL equation:

$$C^\pm = (\pm u, 0,0; \pm 1,0,0), \quad R^\pm = (0, \pm u, 0; 0, \pm 1, 0), \quad P^\pm = (0,0,\pm u; 0,0,\pm 1).$$

In connection with the existence of soliton solutions, we should consider the points $R^\pm$ and $P^\pm$, since they have eigenvalues with a nonzero real part. Solutions of system (2), which go from one equilibrium state to a symmetric state, correspond to stationary topological solitons; solutions which go from a given equilibrium state to the same state correspond to nontopological solitons. At $\nu=0$ there are the following stationary topological solitons: 1) two solutions, which exist for all u, which go from R^+ to R^-, and two solutions which go from R^- to R^+; 2) four solutions which exist at $|u|<u_- = \sqrt{1+\epsilon}-1$, which go from P^+ to P^-, and four corresponding solutions which go from P^- to P^+ (at these values of u, the points P^+ are saddle points). Nontopological solitons exist under the condition $|u|<u_t = \sqrt{1+\epsilon}+1$; at $|u|<u$ they each form four single-parameter families, which are asymptotic families at each point $P^\pm$, separated by the solutions from 2), while at $u_- < |u| < u_+$ they form a single-parameter family of doubly asymptotic trajectories to each point $P^\pm$. Numerical results derived in Ref. 3 point to this structure of stationary solitons; a theory for them is derived in Ref. 4.

We also note the existence at each h, $H(R^\pm) < h < H(P^\pm)$ and $h > H(P^\pm)$ of a single-parameter family of "nonzero vacuum" solitons, which correspond to solutions of system (2) which are asymptotic in the limit $\xi\to\infty$ with respect to a single periodic saddle solution, while in the limit $\xi\to-\infty$ they are asymptotic with respect to a symmetric solution. These periodic solutions arise from the points $R^\pm$ as $h = H(R^\pm)$ is crossed.

At $\nu\neq 0$, system (2) ceases to be integrable, and these one-parameter families of nontopological solitons are destroyed. This result was found numerically in Ref. 3. Analytic results guaranteeing nonintegrability are given here; they are derived through

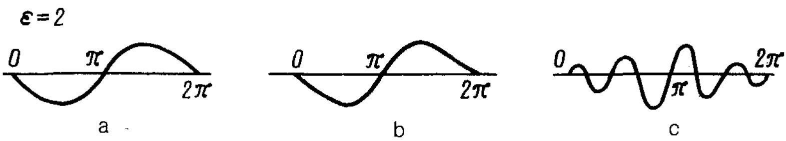

FIG. 1.

the use of an equation itself derived in Ref. 5. We introduce $H_1 = (m \cdot J_0 m)^2$, and we denote by $x(\xi,\theta)$ solutions of system (2) for $\nu = 0$ which are double asymptotic to the point P^+, where θ is a parameter of this family of nontopological solitons. If the function (a Mel'nikov-Arnol'd function)

$$f(\theta) = \int_{-\infty}^{\infty} \{H_1, Q\}\, d\xi \,, \qquad (3)$$

where the integration is carried out along the solution $x(\xi,\theta)$, has a simple zero, then it corresponds to a doubly asymptotic (to P^+) solution of system (2) with $\nu \neq 0$, along which a stable manifold and an unstable manifold intersect in a transversal fashion (without touching) at the level of the Hamiltonian containing the singular point P^+. To evaluate the integral in (3), we need to know the solutions $x(\xi,\theta)$. Corresponding formulas have been furnished by A. I. Bobenko, whom the author thanks. A calculation of the function, f, whose coefficients depend on u, for various values of the parameter u in the intervals $|u| < u_-$ and $u_- < |u| < u_+$ has been carried out numerically. The results for $u_- < |u| < u_+$ are shown in Figs. 1(a)–1(c). These results show that a decay of separatrix surfaces occurs. The number of zeros varies with u. The function f always has zeros at $\theta = 0$, π, because of the symmetry of the formulas for $x(\xi,\theta)$ (as was pointed out to the author by N. E. Kulagin). The zero $\theta = 0$, however, may become a multiple zero; in this case new zeros would appear. The double asymptotic trajectories corresponding to the zeros of the function f are single-circumvention trajectories. By this we mean that, emerging from a singular point, the trajectories arrive at it after a single circumvention of the neighborhood of the unperturbed trajectory. It follows from the results of Ref. 6 that near such a trajectory there exists a countable set of double asymptotic trajectories of arbitrary numbers of circumventions. Consequently, stationary solitons with any prespecified number of hills, corresponding to the circumvention of any single-circumvention solution, exist here. It follows that system (2) is not integrable in the case $\nu \neq 0$ and that there exist stationary solitions of complex profile for values $u_- < |u| < u_+$. In a corresponding way, we could demonstrate the decay of the separatrix surfaces of the saddle point P^+ at $|u| < u_-$ for "nonzero-vacuum" solitons. In that case the nonintegrability follows from Ref. 7.

I wish to thank V. M. Eleonskiĭ, N. E. Kulagin, and Ya. L. Umanskiĭ for stimulating discussions.

[1]V. M. Eleonskiĭ, *Nonlinear Waves. Self-Organization.* Nauka, Moscow, 1983, p. 29.
[2]S. P. Novikov, UMN **37**, 3 (1982).

[3]V. M. Eleonskiĭ *et al.*, Zh. Eksp. Teor. Fiz. **77**, 409 (1979) [Sov. Phys. JETP **50**, 209 (1979)].
[4]L. M. Lerman and Ya. L. Umanskiĭ, *Methods of Qualitative Theory of Differential Equations*, Gor'kovskii Univ., 1983, 3; 1984, 26.
[5]L. M. Lerman and Ya. L. Umanskiĭ, Prikl. Mat. Mekh. **47**, 395 (1983).
[6]R. L. Devaney, in *Bifurcation Theory and Applications in Scientific Disciplines*, Vol. 316, Acad. Sci. New York, 1979, p. 108.
[7]D. V. Turaev and L. P. Shil'nikov, Dokl. Akad. Nauk SSSR **304**, 811 (1989) [Sov. Phys. Dokl. **34**, 311 (1989)].

Translated by Dave Parsons