Two-loop approximation in quantum gravitation

I. L. Bukhbinder, S. D. Odintsov, and O. A. Fonarev
Tomsk Pedagogical Institute

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An effective two-loop action in a multidimensional quantum gravitation, set against a $M_N \times T_K$ background, where M_N is the N-dimensional Minkowski space and $T_K - K$ is a K-dimensional torus, has been calculated for the first time.

The Kaluza–Klein quantum theories are nontrivial physical models in which the gravitational interaction manifests itself at the quantum-mechanical level (see, e.g., Ref. 1). The central theme of these theories is the effective action used to study the spontaneous compactification with allowance for the quantum-gravitational effects.

All the known effective-action calculations in the Kaluza–Klein quantum theories have so far been carried out in the one-loop approximation. It is difficult to go beyond the scope of a single-loop approximation in quantum gravitation (first) because of the complexity of algebraic manipulations, and (second) because new calculation techniques must be developed, since the problem no longer reduces to the customary expression Indet $(\Box + P)$. We will show below on the basis of the Kaluza–Klein quantum theories that both these difficulties can be surmounted.

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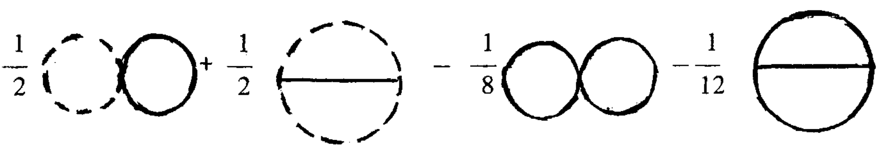
We will calculate effective two-loop action in a multidimensional Einstein quantum gravitation against the background $M_N \times T_K$ $d = N + K \geqslant 4$. In the theory under consideration the action, with allowance for the gauge and the hosts, has the form

$$S = \int d^{N+K} x \left\{ -\frac{1}{2\kappa^2} \sqrt{|g|} R + \frac{1}{2\kappa^2} \chi_A \bar{g}^{AB} \chi_B + \sqrt{|\bar{g}|} \bar{c}^A \chi_A{}^{CE} D_{CE}{}^B [\bar{g} + \kappa h] c_B \right\}. \tag{1}$$

Here $g_{AB} = \bar{g}_{AB} + \kappa h_{AB}$, where $\bar{g}_{AB}$ is the background metric which corresponds to $M_N \times T_K$ and h_{AB} is the quantum gravitation field; $\chi_A \equiv \chi_A^{BC} h_{BC} = \kappa(\delta_A^{(B} \nabla^{C)} - \frac{1}{2}\bar{g}^{BC} \nabla_A) h_{BC}$, $D_{CE}{}^B [g] = 2\delta_{(C}{}^B \widehat{\nabla}_{E)} \bar{c}^A$, c_B are the host fields, and $\widehat{\nabla}_A$ is the covariant derivative constructed on the basis of the background metric. The remaining notation is standard.

The two-loop diagrams which contribute to the effective action are[2]



(2)

Analysis of the propagators reveals that in the calculation of the diagrams (2) in the momentum representation each vertex involves not only integration over the continuous momenta p_α but also summation over $n_1, n_2, \ldots, n_K$.

It is obvious that an explicit calculation of diagrams (2) presupposes a regularization. The ζ regularization can be used for the one-loop calculations in the Kaluza–Klein quantum theories. Its natural extension to the multi-loop case is the generalized operator regularization (see e.g., Ref. 3) which we will use to calculate the momentum integrals.

The results of calculations. We will write the effective action in the form $\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} \ldots$. The two-loop component $\Gamma^{(2)}$ of the effective action will then have the form $\Gamma^{(2)} = \int d^N x V^{(2)}$. We write the expression for $v^{(2)}$ for the odd and even N in the form

$$N = 2\omega + 1, \quad V^{(2)} = \frac{\kappa^2 \pi^{d-2}}{2^{2-K} L_1 L_2 \ldots L_K} Q(d) \left(\frac{2\omega + 1}{2}\right)^2 .$$

Here the solid line corresponds to the gravitation propagator and the dashed line corresponds to the host propagator. The vertices in the diagrams are formed by expanding the action (1) up to and including the fourth-order in the quantum fields h_{AB}; the expressions for these vertices are not given here. The graviton and host propagators are written in the form

$$G(x_1, x_2) = \frac{1}{L_1 L_2 \ldots L_K} \sum_{n_1 = -\infty}^{\infty} \ldots \sum_{n_K = -\infty}^{\infty} \int \frac{d^N p}{(2\pi)^N} e^{i p_A (x_1^A - x_2^A)} G(p)$$

Here $p_A \equiv (p_\alpha, 2\pi/L_a\, n_a)$; $\alpha = 0,1,\ldots, N-1$; $a = 1,2,\ldots,K$; $L_a = 2\pi R_a$, where R_a is the radius of the torus, and $G(p)$ is either a graviton propagator $G_{A_1B_1}{}^{A_2B_2}$ or a host propagator $G_A B$:

$$G_{A_1B_1}{}^{A_2B_2}(p) = -\frac{4P_{A_1B_1}{}^{A_2B_2}}{p^2}, \qquad G_A{}^B(p) = \frac{\delta_A^B}{p^2} \tag{3}$$

$$P_{A_1B_1,A_2B_2} = \bar{g}_{A_1(A_2}\bar{g}_{B_2)B_1} - \frac{1}{d-2}\bar{g}_{A_1A_2}\bar{g}_{B_1B_2},$$

$$p^2 = p_\alpha p^\alpha - \sum_{a=1}^{K}\left(\frac{2\pi}{L_a}n_a\right)^2 \Gamma^2\left(-\frac{2\omega+1}{2}\right)$$
$$\times\left[\zeta_K\left(-\omega-\frac{1}{2}, \frac{1}{2}-\omega\right) + \zeta_K\left(\frac{1}{2}-\omega, -\omega-\frac{1}{2}\right) - \eta_K\left(\frac{1}{2}-\omega, \frac{1}{2}-\omega\right)\right] \tag{4}$$

Here

$$\zeta_K(a,b) = \sum_{n_1,\ldots,n_K,m_1,\ldots,m_K = -\infty}^{\infty} \frac{1}{\left[\left(\frac{n_1}{L_1}\right)^2 + \ldots + \left(\frac{n_K}{L_K}\right)^2\right]^a \left[\left(\frac{n_1+m_1}{L_1}\right)^2 + \ldots + \left(\frac{n_K+m_K}{L_K}\right)^2\right]^b} \tag{5}$$

where η_K (a,b) differs from ζ_K (a,b) because of the presence of the factor $[(m_1/L_1)^2 + \ldots + (m_K/L_K)^2]$ in the numerator. At $a,b \leqslant 1$, the functions ζ_K (a,b) and η_K (a,b) are the analytic continuation and can be explicitly calculated by means of the ζ regularization (see, e.g., Ref. 4). The expression for $Q(d)$ in (4) is

$$Q(d) = \frac{\lambda^2}{9}\left[\frac{d^6}{32} - \frac{d^5}{16} + \frac{41d^4}{32} - \frac{19d^3}{2} + \frac{77d^2}{4} - 15d + \frac{9}{2}\right]$$
$$+ \frac{\lambda^2}{27}\left[-\frac{27d^5}{32} + \frac{219d^4}{16} - \frac{2739d^3}{32} + \frac{827d^2}{4} - \frac{573d}{4} + \frac{27}{4}\right]$$
$$+ \frac{\lambda}{27}\left[\frac{129d^4}{32} - \frac{1395d^3}{8} + \frac{81d^2}{16} - \frac{1503d}{8} + \frac{1059}{4}\right]$$
$$+ \frac{1}{27}\left[-\frac{129d^3}{16} + \frac{123d^2}{16} - \frac{957d}{8} + 174\right], \qquad \lambda = \frac{2}{d-2}$$

2. $N = 2\omega\,,\qquad V^{(2)} = \dfrac{\kappa^2 \pi^{d-2}}{2^{3-K}[(\omega-1)!]^2 L_1 L_2 \dots L_K}\, Q(d)$

$$\cdot \Big\{ \Big[16\ln^2\Big(\frac{\mu}{2\pi}\Big) + 16\Big(\frac{1}{2\omega} + \sum_{l=1}^{\omega-1}\frac{1}{l}\Big)\ln\Big(\frac{\mu}{2\pi}\Big) + 2\Big(\frac{1}{2\omega^2}$$

$$+ \sum_{l=1}^{\omega-1}\frac{1}{l^2}\Big)\Big(1 + \frac{1}{\omega^2} + 2\sum_{S=1}^{\omega-1}\frac{1}{S^2}\Big)\Big]\,[\zeta_K(-\omega, 1-\omega) + \zeta_K(1-\omega, -\omega)$$

$$- \eta_K(1-\omega, 1-\omega)] + \Big[8\ln\Big(\frac{\mu}{2\pi}\Big) + 4\Big(\frac{1}{2\omega} + \sum_{l=1}^{\omega-1}\frac{1}{l}\Big)\Big]\,[\zeta'_K(-\omega, 1-\omega]$$

$$+ \zeta''_K(1-\omega, -\omega) - \eta'_K(1-\omega, 1-\omega) + \frac{1}{\omega}\eta_K(1-\omega, 1-\omega)]$$

$$+ \zeta''_K(1-\omega, -\omega) - \eta'_K(1-\omega, 1-\omega) - \frac{1}{\omega}\eta_K(1-\omega, 1-\omega)\Big\}. \qquad (6)$$

Here μ is an arbitrary mass dimension parameter

$$\zeta_K^{(n)} = \frac{\partial^n}{\partial\alpha^n}\zeta_K(a+\alpha, b+\alpha)\,\Big|_{\alpha=0}\,.$$

Correspondingly, we determine $\eta_K^{(n)}\ (a,b)$. Note that $\Gamma^{(0)} = 0$ against the background we are considering, and $\Gamma^{(1)}$ is given in Ref. 1. Because the background is consistent with the equations of motion, the effective action does not depend on the gauge.

Let us consider the case $N = 4$, $K = 1$. Taking the two-loop components into account, the effective potential is

$$V = \frac{a_1}{L^4} + \frac{\kappa^2}{L^7}\Big(a_2\ln^2\frac{\mu L}{2\pi} + a_3\ln\frac{\mu L}{2\pi} + a_4\Big),$$

where a_1, a_2, a_3, and a_4 are constants. The condition for normalization is the condition under which the induced cosmological constant vanishes. It can be shown that with allowance for $V = 0$, the equation of motion $\partial V/\partial L = 0$ has a solution which determines the compactification radius L. Note that in a single-loop approximation the system of equations $V = 0$, $\partial V/\partial L = 0$ is incompatible with that of Ref. 5, and the bare Λ term must be taken into account. A five-dimensional Einstein gravitation without a bare Λ term thus accounts for a spontaneous compactification when a two-loop approximation is used.

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