Fermion spectrum in magnetic field on lattice with diagonal hops

A. A. Belov, Yu. E. Lozovik, and V. A. Mandel'shtam
Institute of Spectroscopy, Academy of Sciences of the USSR

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A generalization of the Hamiltonian for fermions in a magnetic field on a square lattice is analyzed: $H_0 + \lambda H_1$, where H_0 is the Hamiltonian describing hops to nearest sites, and H_1 is the Hamiltonian of diagonal hops. The density of states is found. There is a nontrivial restructuring of the band structure as the parameter λ is varied.

Over the past few years, the problem of the spectrum of lattice fermions in a magnetic field[1–3] has been revived repeatedly in various branches of physics: incommensurability effects in quasi-1D structures, the quantum Hall effect, high-T_c superconductivity (the mean-field approximation for the RVB model[4]), etc. In connection with high-T_c superconductivity, interest has recently increased in models whose ground state is a chiral spin liquid in which P and T parity is broken. For a spin-1/2 Heisenberg antiferromagnet on a square lattice,

$$H_{AF} = J\sum_{NN} \mathbf{S}_i \times \mathbf{S}_j + \tilde{\lambda} J \sum_{NNN} \mathbf{S}_i \mathbf{S}_j,$$

numerical simulations indicate that the ground state is antiferromagnetic with $\tilde{\lambda} = 0$. It has been shown[5] in the mean-field approximation that a chiral spin state appears only at $\tilde{\lambda} \gtrsim 0.5$. The extreme complexity of the model makes even an approximate Hamiltonian worthy of examination.

The purpose of the present letter is to study the spectrum of the following Hamiltonian on a square lattice:

$$H_{TB} = -\chi \sum_{NN} (e^{i\theta_{ij}} c_i^+ c_j + \text{H.c.}) - \frac{\lambda\chi}{2} \sum_{NNN} (e^{i\theta_{ij}} c_i^+ c_j + \text{H.c.}), \tag{1}$$

This Hamiltonian is a natural generalization of the "flux phase" which corresponds to one of the saddle points in the limit $n \to \infty$ (n is the number of colors).

We consider only the simplest case, that of a uniform magnetic field. We assume that the flux through the plaquette of the lattice is $2\pi\phi$ (in units of the flux quantum), where $\phi = p/q$ is a rational number (the value $\phi = 1/2$ corresponds to the "flux phase").

We know that with $\lambda = 0$ and even q, there is a touching of two bands at zero energy. The effective dynamics near the point of half filling is thus described by a "multiplet" of q relativistic (massless) fermions. Below we consider the spectrum of (1) for all values of λ. It follows from our analysis, in particular, that at $\lambda \geq$, with even q, and at zero energy there is a finite gap between the bands, so the fermions acquire a mass (which depends on λ). After an integration over the massive fermions in the spirit of Ref. 5, we find an effective Chern-Simon Lagrangian with a coefficient of $2q$.

Let us examine the spectrum of Hamiltonian (1). We choose the Landau gauge. We take Fourier transforms and orthonormalize the energy in a suitable way. As a result, we find an equation

$$H\psi_j \equiv (H_0 + \lambda H_1)\psi_j = \epsilon\psi_j , \tag{2}$$

where H_0 is the Harper operator (or the nearly Mathieu operator)

$$H_0\psi_j = 2\cos(\tilde{k}_x + 2\pi\phi j)\psi_j + e^{ik_y}\psi_{j+1} + e^{-ik_y}\psi_{j-1} , \tag{3}$$

$\mathbf{k} = (k_x, k_y)$ is the quasimomentum vector, $k_x = \tilde{k}_x + 2\pi\phi j$, $\tilde{k}_x \in (-\pi/q, \pi/q)$, and H_1 is the "perturbation" operator which describes hops along diagonals of the plaquettes,

$$H_1\psi_j = \cos(\tilde{k}_x + 2\pi\phi(j + \frac{1}{2}))e^{ik_y}\psi_{j+1} + \cos(\tilde{k}_x + 2\pi\phi(j - \frac{1}{2}))e^{-ik_y}\psi_{j-1} \tag{4}$$

The dispersion relation for Eq. (2) is

$$P_\phi(\epsilon, \lambda) = u(\lambda)[\cos(q\tilde{k}_x) + \cos(qk_y)] + v(\lambda)\cos(q\tilde{k}_x)\cos(qk_y), \tag{5}$$

where

$$u(\lambda) = 2^{2-2q} \left| \sum_{k=0}^{q} C_{2q}^{2k}(1+\lambda)^k(1-\lambda)^{q-k} \right| ,$$

$$v(\lambda) = 2^{2-q}\lambda^q, P_\phi(\epsilon, \lambda) = (-1)^{p+q+1}\epsilon^q + \ldots$$

is a polynomial in ϵ of degree q with coefficients which are polynomial functions of λ.

At a fixed value $\phi = p/q$ the spectrum consists of q bands. The density of states in the sth band ($s = 0, \cdots, q-1$) is given by

$$g_s(\epsilon) = \int_{-\pi/q}^{\pi/q} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \delta(\epsilon_s(\tilde{k}_x, k_y) - \epsilon),$$

where $\epsilon_s(\tilde{k}_x, k_y)$ is the sth branch of dispersion relation (5). After several transformations we find

$$g(\epsilon) \equiv \frac{1}{q} \sum_{s=0}^{q-1} g_s(\epsilon) = \frac{1}{\pi^2} \left| \frac{\partial P_\phi(\epsilon, \lambda)}{\partial \epsilon} \right| J(\epsilon), \tag{6}$$

where $J(\epsilon)$ is an elliptic integral whose form varies with the parameters of the problem: $w_1 = -v, w_2 = -u^2/v, w_3 = v - 2u, w_4 = v + 2u$. There are three types of regions along the scale of $P_\phi(\epsilon, \lambda)$, bounded by the parameters w_i, in each of which $J(\epsilon)$ has a separate representation:

$$\text{I. } w_1 < P < w_2. \quad J(\epsilon) = \frac{2}{\sqrt{(w_3 - P)(w_4 - P)}} K\left(\frac{P - w_1}{\sqrt{(w_3 - P)(w_4 - P)}}\right).$$

$$\text{II. } w_2 < P < w_3. \quad J(\epsilon) = \frac{2}{P - w_1} K\left(\frac{\sqrt{(w_3 - P)(w_4 - P)}}{P - w_1}\right). \tag{7}$$

$$\text{III. } w_3 < P < w_4. \quad J(\epsilon) = \frac{1}{\sqrt{v(P - w_2)}} K\left(\sqrt{\frac{(w_4 - P)(P - w_3)}{4v(P - w_2)}}\right).$$

Let us see how the band structure is changed as the parameter λ is varied. At $\lambda = 0$, the spectrum is obviously identical to the spectrum found by Hofstadter,[1] while in the limit $\lambda \to \infty$ the spectrum corresponds, within a scale factor, to the Hofstadter spectrum with a doubled value of ϕ. The range of the parameter λ is broken up into two subranges, $(0,1)$ and $(1, +\infty)$. Within each of these subranges the bands are deformed in a fairly regular way.

In the region $\lambda \in (1, +\infty)$ each band consists of regions of all three types. The band boundaries are found from the equations $P = w_1$ and $P = w_4$. At $P = w_2$ there is a logarithmic singularity. In this region, the parameters w_i are ordered in the following way: $w_1 < w_2 < w_3 < w_4$. At $P = w_3$ the spectrum is continuous.

The point $\lambda = 1$ is "singular." At it, the trajectories $\epsilon_1(\lambda)$, $\epsilon_2(\lambda)$, and $\epsilon_3(\lambda)$ intersect, where $\epsilon_k(\lambda)$ is the solution of the equation $P(\epsilon,\lambda) = w_k$. As a result, the singularities $g(\epsilon)$ at the point $\lambda = 1$ are at the edges of a band, while in the case $\lambda \neq 1$ the singularities are within the bands. The nature of the singularities changes: At the point $\lambda = 1$, the singularities are root singularities. As λ varies from 0 to ∞, all the bands deform in a common, "universal" way. The bands do not overlap. The scaling properties of the spectrum[6] are retained in the course of the deformation. The entire one-parameter family of Hamiltonians $\mathbf{H}(\lambda) = H_0 + \lambda H_1$, with $\lambda \neq 1$, corresponds, as does the nearly Mathieu operator H_0, to a "critical regime." In particular, the spectrum $H(\lambda,\phi)$ is invariant under renormalization-group transformations; the spectrum is singular if ϕ is irrational; etc.

These universality properties of $H(\lambda)$ follow from the symmetry of the classical analog of this system, which is described by the Hamiltonian $H_{\mathrm{CL}} = 2\cos x + 2\cos p + \lambda(\cos(x + P) + \cos(x - p))$, under the interchange $x \leftrightarrow p$. As was shown in Ref. 3, this symmetry leads to a closure of the trajectories of the classical dynamic system and thus to a critical regime of the corresponding quantum-mechanical problem. A more detailed study of the scaling properties of the spectrum of the Hamiltonian $H(\lambda)$ will be published separately.

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