Effective gauge theory of strongly correlated systems

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A $2D$ Hubbard model is analyzed in the limit of strong Coulomb repulsion. An effective gauge theory with a Chern-Simon term and with Dirac fermions is constructed to describe the dynamics of charge carriers. Spin fluctuations are taken into account.

There has recently been an active discussion of a spontaneous breaking of electromagnetic symmetry induced by a topologically massive gauge field[2–9] as one of various possible mechanisms for superconductivity.[1] A question which naturally arises is the microscopic origin of these theories.[2,4,6,8] In the present paper we examine a $2D$ Hubbard model in the limit of a strong Coulomb repulsion. We introduce a spinless charged fermion field and a spinor uncharged boson field, to describe the spin and charge degrees of freedom of the system. Integrating over the spinor fields in the mean-field approximation, and considering only the fluctuations of the constituent gauge fields, we find an effective gauge theory of the Chern-Simons type with Dirac fermions. This theory describes the dynamics of charge carriers.

The original Hamiltonian of the model is

$$H = -t \sum_{\langle i,j \rangle} [C^{+}_{i\sigma} C_{j\sigma} + \text{H.c.}] + J_1 \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_{\langle\langle i,j \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j - \mu \sum_i C^{+}_{i\sigma} C_{i\sigma} \quad (1)$$

The creation (annihilation) operators $C^{+}_{i\sigma}$ ($C_{i\sigma}$) ($\sigma = \uparrow, \downarrow$) are projected onto a subspace which is defined at each site of the $2D$ square lattice by the vectors ($|0\rangle_i$, $|\uparrow\rangle_i$,

$|\downarrow\rangle_i$). We denote by $\langle i,j\rangle$ a summation over nearest neighbors, and by $\langle\langle i,j\rangle\rangle$ a summation over nearest neighbors along a diagonal. Here $\mathbf{S}_i = \frac{1}{2}C_i^+ \vec{\sigma} C_i$, where $\vec{\sigma}$ are the Pauli matrices.

It is convenient to introduce Hubbard operators $X_i^{ab} = |a\rangle_{ii}\langle b|$. These operators satisfy the (anti-) commutation relations of superalgebra SU(2/1) and they can be used to construct coherent states

$$|\xi, Z\rangle = \exp\sum_i[\xi_i X_i^{0\uparrow} + Z_i X_i^{\downarrow\uparrow}]\,|\uparrow\rangle, \tag{2}$$

where Z_i are c-numbers, ξ_i are complex Grassmann variables, and $|\uparrow\rangle = \otimes|\uparrow\rangle$. Following the general approach, we can construct an action for the field theory of our original model (1^i):

$$A = \int_0^\beta d\tau\{\sum_i[\varphi_i^{\sigma+}\dot{\varphi}_i^\sigma + \psi_i^+\dot{\psi}_i - iB_i(\varphi_i^{\sigma+}\varphi_i^\sigma + \psi_i^+\psi_i - 1)] + h(\tau)\}, \tag{3}$$

where the Hamiltonian is

$$h(\tau) = t\sum_{\langle i,j\rangle}[\psi_i^+\psi_j(\varphi_i^\sigma\varphi_j^{\sigma+}) + \text{h.c.}] + J_1\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j + J_2\sum_{\langle\langle i,j\rangle\rangle}\mathbf{S}_i\cdot\mathbf{S}_j - \mu\sum_i(1-\psi_i^+\psi_i), \tag{4}$$

where $\mathbf{S}_i = \frac{1}{2}\varphi_i^+\vec{\sigma}\varphi_i$. The fields $\varphi_i^\sigma(\tau)$ are uncharged boson fields, while the fields $\psi_i(\tau)$ are fermion fields. They describe the spin and charge degrees of freedom, respectively. Varying $B_i(\tau)$ by means of a Lagrange multiplier, we find the relationship $\varphi_i^{\sigma+}\varphi_i^\sigma + \psi_i^+\psi_i = 1$. The Lagrangian is $U(1)$-gauge-invariant. Imposing the gauge condition $\arg\varphi_i^1(\tau) = 0$, and solving the constraint, we find a Lagrangian in terms of the fields $Z_i(\tau)$ and $\xi_i(\tau)$ in (2).

We carry out a Hubbard-Stratanovich transformation and introduce collective coordinates $U_{ij}(\tau)$ and $V_{ij}(\tau)$. The Hamiltonian is

$$\begin{aligned} h(\tau) = &-\frac{2t}{J_1}\sum_{\langle i,j\rangle}[\psi_j^+U_{ji}\psi_i + \psi_i^+U_{ij}\psi_j] + \frac{2t^2}{J_1}\sum_{\langle i,j\rangle}\psi_i^+\psi_i\psi_j^+\psi_j \\ &-\sum_{\langle i,j\rangle}[\frac{2}{J_1}U_{ij}^+U_{ij} - U_{ij}^+\varphi_i^\sigma\varphi_j^{\sigma+} - U_{ij}\varphi_i^{\sigma+}\varphi_j^\sigma] \\ &-\sum_{\langle\langle i,j\rangle\rangle}[\frac{2}{J_2}V_{ij}^+V_{ij} - V_{ij}\varphi_i^\sigma\varphi_j^{\sigma+} - V_{ij}\varphi_i^{\sigma+}\varphi_j^\sigma] - \mu\sum_i(1-\psi_i^+\psi_i). \end{aligned} \tag{5}$$

We are not writing terms of the type $(|\varphi_i^1|^2 + |\varphi_i^2|^2)(|\varphi_j^1|^2 + |\varphi_j^2|^2)$, since they lead to a renormalization of the four-fermion interaction and of the chemical potential. In terms of the spinor fields $\varphi_i^\sigma(\tau)$, the integral is Gaussian. The integration over the fields $U_{ij}(\tau)$ and $V_{ij}(\tau)$ is carried out by the method of steepest descent. We set

$$U_{ij}(\tau) = \rho_1\exp i[\delta_{ij} + B_{ij}(\tau)], \quad V_{ij}(\tau) = \rho_2\exp i[\alpha_{ij} + A_{ij}(\tau)], \tag{6}$$

where ρ_1 and ρ_2 are the parameters used for the variation. The phases are set by the potential of the uniform magnetic field, **B**, which is oriented perpendicular to the lattice. The flux of this field through one plaquette is π: $\delta_{ij} = \int_i \mathbf{B}(\mathbf{x})\cdot d\mathbf{x}$. The phases α_{ij} are[9,10]

$$\alpha(i_x, i_y; i_x + a, i_y + a) = \alpha(i_x, i_y; i_x - a, i_y - a) = \frac{\pi}{2}(-1)^{i_x/a}, \quad (7)$$
$$\alpha(i_x, i_y; i_x + a, i_y - a) = \alpha(i_x, i_y; i_x - a, i_y + a) = -\frac{\pi}{2}(-1)^{i_x/a},$$

where a is the length of an edge of the lattice. The fields $B_{ij}(\tau)$ and $A_{ij}(\tau)$ are gauge fields, which are part of the effective Lagrangian.

The Hamiltonian for the spinor fields $(\varphi_i^\sigma(\tau))$ can be diagonalized [in the case $A_{ij}(\tau) = 0$, $B_{ij}(\tau) = 0$]; the spectrum is[10,11]

$$E^{\pm} = \pm\epsilon(k) = \pm 2[\rho_1^2(\sin^2 k_x a + \sin^2 k_y a) + 4\rho_2^2 \cos^2 k_x a \cos^2 k_y a]^{1/2}, \quad (8)$$

where $-(\pi/a) \leqslant k_x, k_y \leqslant \pi/a$. For the free energy of the spin degrees of freedom we then find

$$F = -\frac{4\rho_1^2}{J_1} - \frac{4\rho_2^2}{J_2} + \frac{2a^2}{\beta}\int d^2k\,(e^{\frac{1}{2}\beta\epsilon} - e^{-\frac{1}{2}\beta\epsilon}). \quad (9)$$

The static-phase equations $\partial F/\partial\rho_1 = 0$, $\partial F/\partial\rho_2 = 0$ have only nonzero solutions.

The energy in (8) has a minimum at isolated points $(k_x^*, k_y^*) = (0,0)$, $(0,\pi/a)$, $(\pi/a,0)$ $(\pi/a,\pi/a)$ if $\rho_1 > 2\rho_2$. Near these points the spectrum is relativistic. In the low-energy limit, the effective Lagrangian for the spinor fields is[10]

$$Z_{\text{eff}} = \sum_{\alpha=1}^{4}\left[\frac{1}{v}\bar{\varphi}_\alpha \tilde{\gamma}_\mu(\partial_\mu - iB_\mu)\varphi_\alpha + m\bar{\varphi}_\alpha\varphi_\alpha\right], \quad (10)$$

where

$$v = 2a\rho_1, \quad m = \frac{2\rho_1}{a\rho_2}; \quad \tilde{\gamma}_1 = \sigma_2; \quad \tilde{\gamma}_2 = \sigma_3; \quad \tilde{\gamma}_3 = v\sigma_3. \quad (11)$$

A distinctive feature here is that the spinor fields $\varphi^\sigma(\mathbf{x})$ are boson fields, but since they appear as Dirac fields in (10) we can find a Chern-Simons term by integrating over them. The coefficient of the kinetic term is proportional to the length of the lattice and contributes nothing in the continuous limit. The gauge field $A_{ij}(\tau)$ also contributes nothing in this limit.

Let us examine the fermion sector. The Hamiltonian of the charged fields is similar to a hopping Hamiltonian in a uniform external magnetic field. As is well known, the spectrum of such a Hamiltonian has zeros at isolated points in the reduced Brillouin zone (there are two such points in the case at hand). Near these points, the spectrum is relativistic, and the low-energy effective Lagrangian contains two Dirac particles with nonzero masses which interact with gauge fields $\mathbf{B}_\mu(x)$ (Ref. 12).

In summary, we find that the effective action describing the dynamics of the charge carriers is

$$A_{\rm eff} = \int d^3x \, [\, \sum_{\alpha=1}^{2} \frac{1}{w} \bar{\psi}_\alpha \hat{\gamma}_\mu (\partial_\mu - iB_\mu)\psi_\alpha + i\frac{1}{2\pi}\epsilon_{\mu\nu\lambda}B_\mu \partial_\nu B_\lambda$$

$$+ \frac{4t^2}{J_1}[(\bar{\psi}\sigma_2\tau_0\psi)(\bar{\psi}\sigma_2\tau_0\psi) - (\bar{\psi}\tau_2\psi)(\bar{\psi}\tau_2\psi)] + \mu\bar{\psi}_\alpha\sigma_2\psi_\alpha\,], \quad (12)$$

where

$$w = \frac{4t\rho_1 a}{J_1}; \quad \hat{\gamma}_1 = \sigma_1; \quad \hat{\gamma}_2 = \sigma_2; \quad \hat{\gamma}_3 = w\sigma_3; \quad \tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

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