Closed equations for interacting gauge fields of all spins

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Closed equations of motion are formulated for interacting gauge fields of all spins in 3 + 1 dimensions.

Recent years have seen definite progress toward the construction of a consistent theory of gauge fields of all spins which are interacting with each other and with gravity in 3 + 1 dimensions. This problem was solved in the cubic approximation in Ref. 1. It was found there that the earlier attempts[2–4] to introduce a gravitational interaction of high-spin gauge fields had failed primarily because this interaction contains negative powers of the cosmological constant and thus ruled out the expansion on a plane background which was used in Refs. 2–4. It was found convenient to describe the dynamics of high spins [5,6,1] in terms of gauge fields corresponding to the high-spin superalgebras which were introduced in Refs. 6–9. These are infinite-dimensional superalgebras, and the corresponding high-spin gauge fields contain infinite chains of fields with spins which increase without bound.

An approach to the high-spin equations of motion which uses an expansion in powers of the curvatures (Weyl 0 forms) was proposed in Refs. 10 and 11. That approach makes it possible to go beyond the cubic approximation in the interaction. All terms of up to second order in the powers of Weyl O forms were derived in Ref. 12. These results correspond to terms of up to fifth order in the interaction at the action level. In the present letter we formulate some completely compatible equations of motion for gauge fields of all spins (in all orders in the interaction). These equations are explicitly invariant in a coordinate-independent way; they contain Einstein's equations; they have all the necessary gauge symmetries; and they reduce to the free equations of the massless fields of all spins in a linearized approximation.

To formulate the dynamics of the gauge fields of all spins, we use the "generating functions" $W(Z,Y;K/x)$ and $B(Z,Y;K/x)$, which are respectively 1 and 0 forms with respect to the space-time coordinates[1] x^ν. The twistor variables $Z:=z_\alpha;\bar{z}_{\dot\alpha})$, $Y:=y_\alpha;\bar{y}_{\dot\alpha})$, (the spinor indices α, β and $\dot\alpha, \dot\beta$ take on the values 1, 2) and the "Klein operators" $K=(k,k)$, which serve as auxiliary variables, satisfy the relations

$$ky_\alpha = -y_\alpha k, \quad kz_\alpha = -z_\alpha k, \quad k^2 = 1, \quad [k,\bar{y}_{\dot\alpha}] = [k,\bar{z}_{\dot\alpha}] = 0; \tag{1}$$

$$ky_{\dot\alpha} = -\bar{y}_{\dot\alpha}\bar{k}, \quad \bar{k}\bar{z}_{\dot\alpha} = -\bar{z}_{\dot\alpha}\bar{k}, \quad \bar{k}^2 = 1, \quad [\bar{k}, y_\alpha] = [\bar{k}, z_\alpha] = 0, \quad [k, \bar{k}] = 0 \tag{2}$$

(the variables z_α, $\bar{z}_{\dot\alpha}$, y_α and $\bar{y}_{\dot\alpha}$ commute).

The primary result of this study is the demonstration that the complete equations of motion of the massless fields of all spins can be reduced to the form

$$dW(Z, Y_0; K) = \int d^4Y_1 d^4Y_2 \exp(-i[\sum_{\substack{n>m\\ n,m = 0-2}} (-1)^{n+m}(Y_m, Y_n)]) \tag{3}$$

$$W(Z, Y_1; K) \wedge W(Z + Y_0 - Y_2, Y_2; K),$$

$$dB(Z, Y_0; K) = \int d^4Y_1 d^4Y_2 \exp(-i[\sum_{\substack{n>m\\ n,m = 0-2}} (-1)^{n+m}(Y_m, Y_n)])$$

$$[W(Z, Y_1; K)B(Z + Y_0 - Y_2, Y_2; K) - B(Z, Y_1; K)W(Z + Y_0 - Y_2, Y_2; K)] \tag{4}$$

$$\frac{\partial}{\partial z^\alpha} W(Z, Y_0; K) = \mu \int d^4Y_1 d^4Y_2 d^4U \delta(\bar{u})$$

$$\times \exp(-i[\sum_{\substack{n>m\\ n,m = 0-2}} (-1)^{n+m}(Y_m, Y_n) + 2\sum_{n=0}^{2} (-1)^n(U, Y_n)]) \tag{5}$$

$$\times \{\nabla_\alpha(z + y_0 + u, y_0 + u) W((-z - 2y_0 - 2u; \bar{z}), Y_1; K)$$

$$-\nabla_\alpha(z + y_0 + u, y_2 - u) W(Z, Y_1; K)\} B((-z - y_0 - y_2; \bar{z} + \bar{y}_0 - \bar{y}_2), Y_2; K)k$$

$$\frac{\partial}{\partial \bar{z}^{\dot\alpha}} W(\ldots) = \ldots, \quad \frac{\partial}{\partial z^\alpha} B(\ldots) = \ldots, \quad \frac{\partial}{\partial \bar{z}^{\dot\alpha}} B(\ldots) = \ldots. \tag{6}$$

Equations (6) were found from Eqs. (5) by means of the substitution $W \to B$ and/or the replacement of spinors with a dot by spinors without a dot, and vice versa, with the simultaneous substitution $\mu\leftrightarrow\bar{\mu}$ (μ is an arbitary complex deformation parameter). We are using the notation

$$(Y_n, Y_m) = y_{\alpha n} y_m^{\alpha} + y_{\dot{\alpha} n} \bar{y}_m^{\alpha}, \tag{7}$$

$$\nabla_\alpha (z, y) = (y_\alpha - z_\alpha) \int_0^1 d\beta \beta \delta (\beta z + (1-\beta) y), \tag{8}$$

where $\delta(y)$ means a two-dimensional Dirac δ-function of spinor argument y_α.

Equation (3) may be thought of as a zero-field equation for determining an infinite-dimensional Lie superalgebra[2] g. In this case, Eq. (4) becomes the condition for covariant constancy of the O forms which take on values in the associated representation of g. These equations become dynamically nontrivial by virtue of Eqs. (5) and (6). To demonstrate this point, we will show that Eqs. (3)–(6) reproduce the high-spin equations of motion of Ref. 11 in the first order of an expansion in Weyl O forms, which corresponds to an expansion in powers of the parameters μ and $\bar{\mu}$.

In zeroth order in μ and $\bar{\mu}$ we find $(W(Z,Y;K) = \omega(Y;K)$ and $B(Z,Y;K) = C(Y;K)$ from Eqs. (5) and (6), where ω and C are respectively the 1 and 0 forms which were used in Refs. 10 and 11. Solving Eqs. (1) in first order in μ, we can easily show that

$$\begin{aligned} W(Z, Y_0; K) = {} & \omega(Y_0; K) + \int d^4 Y_1 d^4 Y_2 d^4 U \\ & \times \exp\Big(-i\Big[\sum_{\substack{n>m \\ n,m=0-2}} (Y_m, Y_n) + 2\sum_{n=0}^{2} (-1)^n (U, Y_n)\Big]\Big) \\ & \times \omega(Y_1; K) C(Y_2; K) \big(\mu k \delta(\bar{u}) \Delta(z + y_0 + u, y_0 + u, y_2 - u) \\ & + \bar{\mu} \bar{k} \delta(u) \Delta(\bar{z} + \bar{y}_0 + \bar{u}, \bar{y}_0 + \bar{u}, \bar{y}_2 - \bar{u})\big) + O(\mu^2, \mu\bar{\mu}, \bar{\mu}^2), \end{aligned} \tag{9}$$

where the function $\Delta(a,b,c)$, of three spinor arguments, was introduced in Ref. 11 and can be described by

$$\begin{aligned} \Delta(a, b, c) = {} & \int d\beta_1 d\beta_2 d\beta_3 \theta(\beta_1) \theta(\beta_2) \theta(\beta_3) \delta\Big(1 - \sum_{n=1}^{3} \beta_n\Big) \\ & \times (a_\alpha b^\alpha + b_\alpha c^\alpha - a_\alpha c^\alpha) \delta(\beta_1 a + \beta_2 b + \beta_3 c). \end{aligned} \tag{10}$$

To derive (9), it is sufficient to consider the (easily verifiable) relation

$$\frac{\partial}{\partial z^\alpha} \Delta(z, y, x) = \nabla_\alpha(z, x) - \nabla_\alpha(z, y). \tag{11}$$

It is not difficult to verify that the substitution of (9) into (3) with $Z = 0$ leads to precisely the equations of motion of Ref. 11. The equations of Ref. 11 or C can be derived in a corresponding way.

We wish to stress that the most important properties of system (3)–(6) are its formal compatibility (the symmetry of the second derivatives) and the fact that it correctly reproduces the free high-spin equations. We proved this second property by reducing Eqs. (3)–(6) to the Eqs. of Ref. 11 (the corrections for higher powers of μ and $\bar{\mu}$ affect the terms with an interaction). The first property can be verified directly. The compatibility of Eqs. (3)–(6) guarantees, in particular, that once we have solved Eqs. (5) and (6), which are algebraic equations with respect to the space-time coordinates, Eqs. (3) and (4) with $Z \neq 0$ will have no information beyond that contained in these equations with $Z = 0$.

An important property of Eqs. (3)–(6) is that they remain compatible if all the W's and B's take on values in an arbitrary associative algebra (a test of the compatibility requires only the associativity of the product law of W and B). As was emphasized in Ref. 10, this property makes it possible to examine various systems of fields with nontrivial internal symmetries (see Ref. 9 for a complete list of such systems).

We have presented only the most compact form of the high-spin equations, which is not itself the most convenient for analyzing consistent truncations (automorphisms) or conditions for a Hermitian nature. Other possible forms of the high-spin equations are related to Eqs. (3)–(6) by nonlinear replacements of the variables and are dynamically equivalent to them, at least in perturbation-theory terms. We cannot go into this matter in more detail here, but we would like to point out that there is a modification of Eqs. (3)–(6) which is explicitly invariant under an antiautomorphism ρ and the Hermitian conjugation law t of Ref. 10. Therefore, all the qualitative results of Refs. 9–11 remain in force in all orders in the interaction.

[1)] In other words, $W = dx^{\nu} W_{\nu}(...)$. From here on we will omit the dependence on the space-time coordinates x_{ν}. The use of the formalism of an external algebra of differential forms guarantees that the entire treatment will be invariant in a coordinate-independent way. The external differential $d = dx^{\nu}(\partial/\partial x^{\nu})$ is the only operator which acts on the coordinates here.

[2)] Using Berezin operator symbols[13,14] (see also Ref. 7), one can easily verify that g is the algebra of all possible functions of the operators $\hat{Y}\hat{Z}\hat{K}$ which obey the relations $[\hat{y}_{\alpha},\hat{y}_{\beta}] = 2i\epsilon_{\alpha\beta}$, $[\hat{z}_{\alpha},\hat{y}_{\beta}] = -i\epsilon_{\alpha\beta}$, $[\hat{z}_{\alpha},\hat{z}_{\beta}] = 0$, $\{\hat{k},\hat{y}_{\alpha}\} = \{\hat{k},\hat{z}_{\alpha}\} = 0$, $k^2 = I$ [it is assumed here that the generators $(\hat{y}_{\alpha},\hat{z}_{\dot{\alpha}}, k)$ obey corresponding conditions; the sets $(\bar{y}_{\alpha}),\bar{z},k)$ and $(\hat{y}_{\dot{\alpha}},\bar{z}_{\dot{\alpha}},k)$ are mutually conjugate and mutually commutative]. The right side of (3) describes the product of symbols in a basis in which the operators $\hat{Y}$ are on the right, and the operators $\hat{Z}$ are on the left; it is assumed here that the operators $\hat{Y}$ are ordered in a completely symmetric way. Algebras of this type were discussed in the context of high-spin theory in the leading dimensonalities in Ref. 15 and in the context of conformal high-spin theories in $3+1$ dimensons in Ref. 16.

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