New exact solution of Einstein's equations for the gravitational field of a stationary axisymmetric mass

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An exact, asymptotically planar solution of the vacuum Einstein's equations is derived to describe the external gravitational field of a rotating axisymmetric mass. This solution has a static Schwarzschild limit. It contains the Kerr metric as a special case.

An asymptotically planar, stationary solution of the vacuum Einstein's equations which has the Schwarzschild metric as its static limit and which differs from the Kerr metric[2] was derived for the first time in Ref. 1. In the present letter we consider a new axisymmetric solution, which generalizes the solution of Ref. 1 and which also contains the Kerr metric as a special case. Because of an additional parameter, the quadrupole moment Q of the source of the gravitational field can be arbitrary in our solution; it need not depend in a strictly defined way on the total mass M and the angular momentum J (for the Kerr solution, the relationship $Q = J^2/M$ holds[3]). This new metric has an event horizon.

The solution, which we derived by a nonlinear superposition method (Ref. 4, for example), is determined by a complex Ernst potential[5]

$$\epsilon = \frac{R_+ + R_- - 2k}{R_+ + R_- + 2k} \frac{A_-}{A_+};$$

$$A_\mp = 16r_+^2 r_-^2 - \alpha^2 (r_+ + r_- \mp 2kl)(r_+ + r_- \pm 2kl)^3 ab$$

$$\mp 4i\alpha(r_+ + r_- \pm 2kl)[r_+^2 (r_+ - r_- \mp 2kl)a + r_-^2 (r_+ - r_- \pm 2kl)b];$$

$$a = \frac{\rho^2 + (z-k)(z-kl) + r_- R_-}{\rho^2 + (z+k)(z-kl) + r_- R_+}; \qquad b = \frac{\rho^2 + (z+k)(z+kl) + r_+ R_+}{\rho^2 + (z-k)(z+kl) + r_+ R_-}; \tag{1}$$

$$R_\pm^2 = \rho^2 + (z \pm k)^2; \quad r_\pm^2 = \rho^2 + (z \pm kl)^2,$$

where ρ and z are the canonical Weyl-Papapetru coordinates, and α, k, and l are real constants.

The corresponding unknown functions, f, γ, and ω in the Papapetru metric

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \tag{2}$$

which describes stationary axisymmetric gravitational fields, can be constructed from the known ϵ by the method developed in Refs. 6–8. The results are the following expressions:

$$f = \frac{4(R_+ + R_- - 2k)B}{(R_+ + R_- + 2k)C}; \qquad \exp(2\gamma) = \frac{(R_+R_- + \rho^2 + z^2 - k^2)B}{32(1-\alpha^2)^2 R_+R_- r_+^4 r_-^4};$$

$$\omega = \frac{4\alpha kl}{\alpha^2 - 1} - \frac{\alpha(R_+ + R_- + 2k)(r_+ + r_- - 2kl)D}{(R_+ + R_- - 2k)B};$$

$$B = [4r_+^2 r_-^2 - \alpha^2 (r_+r_- + \rho^2 + z^2 - k^2 l^2)^2 ab]^2 - 16\alpha^2 k^2 l^2 \rho^2 (r_+^2 a + r_-^2 b)^2;$$

$$C = [8r_+^2 r_-^2 - \alpha^2 (r_+r_- + \rho^2 + z^2 - k^2 l^2)(r_+ + r_- - 2kl)^2 ab]^2$$

$$+ 4\alpha^2 (r_+ + r_- - 2kl)^2 [(r_+ - r_- + 2kl)r_+^2 a + (r_+ - r_- - 2kl) r_-^2 b]^2;$$

$$D = 8r_+^3 r_-^3 [(r_+ - r_- - 2kl)r_- b - (r_+ - r_- + 2kl)r_+ a]$$

$$- \alpha^2 (r_+r_- + \rho^2 + z^2 - k^2 l^2)(r_+ + r_- - 2kl)^2 \; [(r_+ - r_- - 2kl)r_-^3 b - (r_+ - r_- + 2kl)r_+^3 a]ab. \quad (3)$$

Expressions (3) and (1) completely determine the new stationary metric. Its static limit (which is reached by setting $\alpha = 0$) is the spherically symmetric Schwarzschild metric.

Using the Reduce 3.3 analytic calculation system, we have calculated the first four multipole moments M_i and J_i by the method of Geroch[9] and Hansen.[10] These moment characterize the distributions of the mass and angular momentum of the source, respectively. The expressions are

$$M_0 \equiv M = k(1 + 2\alpha^2 l - \alpha^2)(1-\alpha^2)^{-1}; \; M_1 = M_3 = 0;$$

$$M_2 \equiv Q = 2\alpha^2 k^3 l[(\alpha^4 + 6\alpha^2 - 3)l^2 - 2(\alpha^2 + 2)(\alpha^2 - 1)l + (\alpha^2 - 1)^2](\alpha^2 - 1)^{-3};$$

$$J_0 = J_2 = 0; \quad J_1 \equiv J = -2\alpha k^2 l[(3\alpha^2 - 1)l - 2\alpha^2 + 2](\alpha^2 - 1)^{-2};$$

$$J_3 = 2\alpha k^4 l^2 [(5\alpha^6 + 11\alpha^4 - 9\alpha^2 + 1)l^2 - 2(5\alpha^2 - 1)(\alpha^4 - 1)l$$

$$+ (\alpha^2 - 1)^2 (5\alpha^2 + 1)](\alpha^2 - 1)^{-4}, \quad (4)$$

It follows immediately from the vanishing of the monopole term J_0 that metric (3) is asymptotically planar.

Let us examine some particular cases of this solution.

1) With $l = 1$ we obtain the Kerr metric. By introducing $p = (1-\alpha^2)/(1+\alpha^2)$, $q = 2\alpha/(1+\alpha^2)$, $p^2 + q^2 = 1$, we can easily put it in standard form. With $l = 1$, we find from (4) the special relationship among the multipole moments M_2, J_1, and M_0 which has been mentioned previously.

2) The case $l = -1$ corresponds to the two-parameter solution derived in Ref. 1.

3) With $\alpha = 0$ (without a rotation), there is a transition to the Schwarzschild metric, and the total mass M_0 becomes equal to k. The fact that this transition occurs even when the "quadrupole" parameter l is zero reflects a fact which is obvious from the physical standpoint: A spherically symmetric source which is initially at rest will unavoidably be deformed when it rotates.

4) Finally, with $k = 0$ (the total mass is zero), this space–time becomes planar.

In contrast, this solution, in general, describes the external gravitational field of a rotating source with the total mass, angular momentum, and quadrupole moment given by (4).

An important physical property of metric (3) is that it has an event horizon, which is specified by the hypersurface

$$R_+ + R_- - 2k = 0 \tag{5}$$

(this assertion can be proved in a straightforward way by a method analogous to that of Ref. 11). The horizon is regular in the case of the Kerr solution ($l = 1$) and in the case of the Schwarzschild solution ($\alpha = 0$ or $l = 0$). At $|l| > 1$ the horizon has only a single singularity, which is on the symmetry axis [in the spheroidal coordinates $r = M + (R_+ + R_-)/2$, $\cos\vartheta = (R_+ - R_-)/(2k)$, it corresponds to the point $r = M + k$, $\vartheta = 0$]. For $|l| < 1$, $l \neq 0$, two more singularities, which are annular and correspond to the values $\cos\vartheta = \pm l$, appear on the horizon. In the particular case of the solution of Ref. 1, with $l = -1$, only the two bands $\cos\vartheta = \pm 1$ are singular. In general, it is difficult to calculate the area of hypersurface (5), but in the case $l = -1$ calculations lead to the compact expression

$$A = 16\pi k^2 (1 - \alpha^2)^{-1}, \tag{6}$$

which in the case $\alpha = 0$ becomes the known result for the area of the horizon of a Schwarzschild metric.

The other singularities of the solution at $l \neq 1$ are determined by the equations $R_+ + R_- + 2k = 0$ and $C = 0$. The real roots of the second equation lie on the surface of the stationarity limit, $B = 0$. It also follows from Robinson's theorem that not all of them are concealed below the horizon; some of them should be in the external region.

This metric is a Petrov type-I metric, degenerating into a type D only for the parameter values $\alpha = l = 0$ and $l = 1$. It is good illustration of Ernst's foresight[13] regarding the possible major physical significance of such metrics.

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