Quantization of 3D particle with twisting: Toward a relativistic theory of an anyon

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A quantum particle with twisting in a pseudo-Euclidean 3D space is characterized by an unquantized spin (it is an anyon) and by an infinite set of internal excitations.

Particle and string theories in which the action contains high-order derivatives of dynamic quantities have recently attracted considerable interest.[1,2] For a 3D point particle there exists a special invariant of this type: the twisting

$$T = \int \frac{\epsilon^{\mu\nu\lambda} \dot{x}_\mu \ddot{x}_\nu \dddot{x}_\lambda}{[\dot{x}, \ddot{x}]^2} \sqrt{\dot{x}^2}\, d\tau . \tag{1}$$

A quantum theory of a particle with twisting has been constructed for the Euclidean case by means of a path integral[3] (the measure of the integration was discussed in detail in Ref. 4). For certain values of the coefficient with which (1) appears in the action, a transmutation of the spin occurs.

In the present letter we consider a particle with twisting in a pseudo-Euclidean space with the action

$$S = -m \int \sqrt{\dot{x}^2}\, d\tau + cT, \tag{2}$$

where $x_\mu(\tau)$, $\mu = 0, 1, 2$, is a timelike path, and $c > 0$. Here are our results. A canonical Dirac quantization[5] leads, after a redefinition of the Poisson brackets, to the spin algebra of the SU(1,1) group. The wave function of a particle satisfies the equation

$$(k^\mu S_\mu - cm)\psi = 0, \tag{3}$$

where k^μ is the energy-momentum operator, and S_μ are the generators of the SU(1,1) representation (more precisely, of its universal covering) with $S^2 = c(c-1)$. This equation describes a particle with an unquantized spin c (the relativistic version of an anyon[6]). Since the representation is infinite-dimensional, the particle is characterized by an infinite spectrum of internal excitations. In the Euclidean version, the parameter c is quantized, and Eq. (3) becomes a Dirac equation with a finite spectrum.

For a canonical formulation of theory (2), we introduce timelike and spacelike unit vectors e_μ and n_μ such that $\dot{x}_\mu = q e_\mu$, $\dot{e}_\mu = Q n_\mu$, and we switch to the action

$$S' = \int d\tau \big(-mq + c\epsilon^{\mu\nu\lambda} e_\mu n_\nu \dot{n}_\lambda - k^\mu(\dot{x}_\mu - q e_\mu) - p^\mu(\dot{e}_\mu - Q n_\mu) + \alpha(e^2 - 1) + \beta(n^2 + 1)\big), \tag{4}$$

where $k_\mu, p_\mu, \alpha, \beta$ are Lagrange multipliers. We see that (x_μ,k_μ) and (e_μ,p_μ) are canonical pairs.

The complete system of constraints (of the first and second kinds) for the case of a nonvanishing curvature $\zeta = Q/q \neq 0$ is

$$\alpha = \pi_\alpha = 0,\ \beta = \pi_\beta = 0,\ Q - c^{-1} q \epsilon^{\mu\nu\lambda} e_\mu n_\nu k_\lambda = 0,\ \pi_Q = 0,$$

$$n^2 + 1 = 0,\ e^\mu n_\mu = 0,\ k^\mu n_\mu = 0,\ \pi_n^\mu + c\epsilon^{\mu\nu\lambda} e_\nu n_\lambda = 0,$$

$$e^2 - 1 = 0, \quad p = 0,$$

$$\pi_q = 0 \tag{5}$$

$$k^\mu e_\mu - m = 0, \tag{6}$$

where π represents canonical momenta. Using this system, we can eliminate (λ,π_λ), (μ,π_μ) and (Q,π_Q). In addition, we can express (n_μ,π_n^μ) in terms of the remaining variables, but since n_μ and π_n^μ are nonzero, the redefinition of the Poisson brackets is nontrivial in this case. Let us calculate a canonical 2-form on the subspace determined by the constraints in the parametrization $e_\mu = (\cosh\theta, \sinh\theta\cos\varphi, \sinh\theta\sin\varphi)$:

$$dp^\mu \wedge de_\mu + d\pi_n^\mu \wedge dn_\mu = -c \sinh\theta d\theta \wedge d\varphi , \tag{7}$$

Hence $\{\theta,\varphi\} = -(c\sinh\theta)^{-1}$ or, equivalently,

$$\{e_\mu, e_\nu\} = -c^{-1} \epsilon_{\mu\nu\lambda} e^\lambda . \tag{8}$$

[In a Euclidean space, Eq. (7) would be replaced by $c \sin\theta d\theta \wedge d\varphi$.] We thus conclude that the particle is described by a Hamiltonian $H = q(m - k^\mu e_\mu)$ with constraints of the first kind as in (5) and in (6) and with algebra (8) for the components of the unit vector e_μ.

It can be shown that the classical path of a particle is a helix. The energy-momentum k^μ and the angular momentum M^μ are conserved. The angular momentum is given by

$$M^\mu = \epsilon^{\mu\nu\lambda}(x_\nu k_\lambda + e_\nu p_\lambda + n_\nu \pi_{n\,\lambda}) = \epsilon^{\mu\nu\lambda} x_\nu k_\lambda + c e^\mu , \tag{9}$$

i.e., the particle acquires a spin c along the direction of the velocity. In the classical theory, the relation $k^2 = m^2 - c^2\zeta^2$ follows from the system of constraints, where ζ is the constant curvature of the path. This relation was derived by a different method in Ref. 7, and it was suggested that it implies the presence of a tachyon in the quantum theory. We will see below that this is not the case.

We represent the phase space of the system as the direct product of the planar space of the variables (x,k) and a 2D hyperboloid (or a sphere, in the Euclidean case). The quantization on the hyperboloid is carried out with the help of coherent states of the SU(1,1) group.[8] The quantities $S_\mu = c e_\mu$ are the generators of the group, and constraint (6) becomes Eq. (3). We assume that the state vector ψ belongs to a unitary representation whose basis vectors are specified by a nonnegative integer n:

$$S_0 \mid c, n\rangle = (c + n) \mid c, n\rangle. \tag{10}$$

The mass spectrum is then

$$E_n = \frac{cm}{c+n}, \qquad n = 0, 1 \ \ldots \ . \tag{11}$$

This spectrum is not cut off, even with integer and half-integer values of c, in which case a Euclidean quantization is possible.

In summary, we have shown that quantum theory (2) describes a relativistic particle with an unquantized spin c and with an infinite set of internal excitations in a 3D pseudo-Euclidean space. It would seem possible to construct a second-quantized version of the theory by replacing ψ in (3) by an operator function.

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