Analogy between initial stages of turbulence nucleation and electrical explosion of conductor

N. B. Volkov and A. M. Iskol'dskiĭ
Institute of Electrophysics, Ural Branch of the Academy of Sciences of the USSR

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An analogy is established between the initial stages of the nucleation of turbulence and the electrical explosion of a conductor. A model of the Lorenz type [E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963)] is proposed. The qualitative behavior of the solution is analyzed, and a comparison is made with experiment.

The pulsed heating of a conductor by an electric current results in a destruction of the conductor in an explosive manner (an electrical explosion).[1,2] During an electrical explosion, the conductor becomes stratified, and a low-temperature plasma with a condensed disperse phase forms. It has been established experimentally[1,2] that in the heating stage, which is substantially longer than the actual explosion stage, the diameter of the conductor varies slowly. A small-scale structure, reminiscent of the thread of a screw, forms at the beginning of the explosion, according to x-ray photography.[2] On this basis we have hypothesized that the initial stage of an electrical explosion has an analog in the stage of the nucleation of turbulence in an incompressible liquid (the formation of Bénard rollers and cells[3,4] and Taylor vortices[4,5]). Let us explain.

In the heating stage, it is sufficient to use the approximation of an incompressible conducting liquid and to consider the perturbation of the external force by the thermal expansion of the conductor; the coefficients in the MHD equation can be assumed constant. In this case the MHD equations become

$$\operatorname{div}\mathbf{u} = 0; \tag{1a}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\vec{\nabla})\mathbf{u} = -\frac{1}{\rho_0}\vec{\nabla}P + \frac{1}{4\pi\rho_0}\left(1 + \frac{\delta\rho_0}{\rho_0}\right)[\operatorname{curl}\mathbf{H},\mathbf{H}] + \nu\Delta\mathbf{u}; \tag{1b}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u}\vec{\nabla})T = \chi\Delta T + \frac{\nu_m}{4\pi C_v}(\operatorname{curl}\mathbf{H})^2; \tag{1c}$$

$$\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{u}\vec{\nabla})\mathbf{H} = (\mathbf{H}\vec{\nabla})\mathbf{u} - \nu_m \operatorname{curl}\operatorname{curl}\mathbf{H}, \tag{1d}$$

where ρ_0, $\mathbf{u}$, T, and $\mathbf{H}$ are respectively the unperturbed density, the velocity, the temperature, and the magnetic field; $\delta\rho = \alpha\rho_0(T - T_0)$ is the density perturbation due to thermal expansion; α is the thermal expansion coefficient; T_0 is the initial temperature; $\nu = \eta/\rho_0$ is the kinematic viscosity (η is the shear viscosity); $\nu_m = C^2(4\pi\sigma)^{-1}$ is the magnetic viscosity (σ is the electrical conductivity); and $\chi = \lambda / C_v$ is the thermal diffusivity.

We direct the z axis along the axis of the conductor, and we make use of the azimuthal symmetry, setting $\mathbf{u} = \{u_r(r,z), 0, u_z(r,z)\}$ and $\mathbf{H} = \{0, H(r,z), 0\}$. From Eq. (1a) we then find $u_r = -\partial\psi/\partial z$, $u_z = \partial(r\psi)/r\partial r$, $\psi(r,z,t)$: a potential function. Equation (1d) has an unperturbed steady-state solution $H_1(r) = \Delta H r/r_0$, where $\Delta H = 2I/Cr_0$, and $I =$ const and $r_0 =$ const are respectively the current and the radius of the conductor. The unperturbed solution of Eq. (1c) is not a steady-state solution: $T(t) = T_0 + \Delta T \nu_m t r_0^{-2}$, where $\Delta T = (\Delta H)^2/C_v$. We restrict the discussion to the unperturbed solution of Eq. (1c), and we set $H(r,z,t) = H_1(r) + h(r,z,t)$. We then find a system of equations which is similar, aside from the replacement of a Cartesian coordinate system by a cylindrical one and the replacement of the heat conduction equation by the equation for the diffusion of a magnetic field, to the system of Saltzman equations[6] in the theory of the Bénard effect:

$$\frac{\partial(\Delta\psi - \psi r^{-2})}{\partial t} = -\frac{\partial(\psi, \Delta\psi - \psi r^{-2})}{\partial(r,z)} + R\left(1 + R_1 \frac{\nu_m t}{r_0^{-2}}\right)\frac{\nu_m \nu}{\Delta H r_0^3}\frac{\partial h}{\partial z}$$

$$+ \nu[\Delta(\Delta\psi - \psi r^{-2}) - r^{-2}(\Delta\psi - \psi r^{-2})], \tag{2a}$$

$$\frac{\partial h}{\partial t} = -\frac{\partial(\psi, h)}{\partial(r,z)} + \frac{\Delta H}{r_0}\frac{\partial\psi}{\partial z} + \nu_m\left(\Delta h - \frac{h}{r^2}\right). \tag{2b}$$

Here $\partial(a,b)/\partial(r,z) = [\partial(ra)/r\partial r](\partial b/\partial z) - (\partial a/\partial z)[\partial(rb)/r\partial r]$, Δ is the Laplacian, $R = (\Delta H)^2 r_0^2/2\pi\rho_0\nu_m\nu = V_A^2 r_0^2(\nu_m\nu)^{-1} = Pe_m^2\sigma^{-1}$ is the Rayleigh number, $V_A = \Delta H(2\pi\rho_0)^{-1/2}$ is the Alfvén velocity, $Pe_m = V_A r_0 \nu_m^{-1}$ is the magnetic Peclet number, $\sigma = \nu\nu_m^{-1}$, $R_1 = \alpha(\Delta H)^2(\pi C_v)^{-1} = (\alpha C_0^2 C_\rho^{-1})(\Delta H)^2(\pi\rho_0 C_0^2)^{-1} = 2\Gamma_0 M_A^2$, $M_A = V_A C_0^{-1}$ is the magnetic Mach number, and Γ_0 is the Grüneisen parameter. Equation (2a) shows that structures can form even in the absence of a thermal expansion. Physically, this result corresponds to the nucleation of MHD instabilities.

Following Lorenz,[7] we assume free boundary conditions for the perturbations: $h(0,z,t) = h(r_0,z,t) = \psi(0,z,t) = \psi(r_0,z,t) = 0$. Retaining the less significant terms in the Fourier representation of ψ and h, we use the substitution

$$\psi = 2^{1/2}\frac{1 + \left(\frac{\pi k}{k_1}\right)^2}{k}\nu_m X(t)\sin\left(\frac{\pi k z}{r_0}\right)J_1\left(\frac{k_1 r}{r_0}\right), \tag{3a}$$

$$h = \frac{R_c\Delta H}{\pi R}\left\{2^{1/2}Y(t)\cos\left(\frac{\pi k z}{r_0}\right)J_1\left(\frac{k_1 r}{r_0}\right) - Z(t)J_1\left(\frac{2k_1 r}{r_0}\right)\right\}. \tag{3b}$$

In (3a) and (3b), $k_1 = 3.83171$ is a zero of the Bessel function $J_1(x)$, $R_c = 64k_1^2\pi^2[b^2(4-b)]^{-1}$ is the critical Rayleigh number, and $b = 4[1+(\pi k k_1^{-1})^2]^{-1}$. As a result, we find a system of ordinary differential equations for the amplitudes $X(t)$, $Y(t)$, and $Z(t)$:

$$\dot{X} = -\sigma X + (1 + 0.5b\Gamma_0 M_A^2 \tau k_1^{-2})\sigma Y, \tag{4a}$$

$$\dot{Y} = -\pi k_1^{-1} XZ + (\pi k_1^{-1})^2 rX - Y, \tag{4b}$$

$$\dot{Z} = -\pi k_1^{-1} XY - bZ, \tag{4c}$$

where $\tau = 4k_1^2 b^{-1} \nu_m t r_0^{-2}$ is a dimensionless time, and $r = RR_c^{-1}$.

If we ignore the time-dependent term in (4a) and use $\pi k_1^{-1} \approx 1$, we find that Eqs. (4a) and (4b) become the same as the first two equations of the famous Lorenz model,[7] which is regarded in the literature as a possible model for the nucleation of turbulence. Equation (4c) has a minus sign instead of a plus sign in front of XY.

The critical Rayleigh number becomes infinite ($R_c \to \infty$) as $b \to 0$ ($b \to 4$), so it has a minimum which corresponds to $b = 8/3$: $R_{c,\min} = 978.08144$. The minimum value of R_c determines the length scale (l) of the structure along the z axis: $l = r_0 k^{-1} = r_0 \pi k_1^{-1} b^{1/2}(4-b)^{-1/2} = 1.15931 r_0$. Correspondingly, we have $r = 1.0224 \times 10^{-3} R = 6.5088 \times 10^{-4} I^2 c^{-2} (\rho_0 \nu_m \nu)^{-1}$.

In Eq. (4a) we ignore the term which reflects thermal expansion, and we analyze the resulting system of equations:

$$\dot{W} = F(W), \qquad W = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{5}$$

Fixed points of system (5) are determined from the condition $F(W) = 0$: $W_1 = 0$, $W_{2,3} = \{\pm 1.221 \times [b(1-0.672r)]^{1/2};\ \pm 1.221\ [\frac{8}{3}(1-0.672r)]^{1/2};\ -1.221(1-0.672r)\}$. We see that system (5) has no fixed points at $r > r_* = 1.488$. Analysis of the stability of the solution shows that at $r < r_*$ the fixed points are stable. The fixed point corresponding to $r = r_*$ is a stability boundary.

The coordinates of the fixed points of the Lorenz model are $W_1 = 0$, $W_{2,3} = \{\pm [b(r-1)]^{1/2};\ \pm [b(r-1)]^{1/2};\ r-1\}$. The critical point of the Lorenz model ($r = R_* = R_c$) is also a stability boundary. We see that there are stable steady-state solutions at $R > R_*$ in the Lorenz model. The transition from a point $W \approx 0$ to an arbitrary point in the phase space occurs through an adiabatic sequence of stable states. A bifurcation occurs at the critical point; in the literature, it is called a "supercritical bifurcation." In our own case, the bifurcation is subcritical, since there are no steady-state solutions near the critical point under the condition $R > R_*$. Any trajectory of the system in the phase space at $R > R_*$ takes the system away from the point $W \approx 0$ off to infinity over a finite time. Such a transition is called "explosive."[5] The term reflecting the thermal expansion substantially reduces the time of the explosive transition. Physically, this effect corresponds to the nucleation of a thermal instability.

This qualitative behavior does not contradict the experiments of Ref. 2, where it was shown that a copper conductor 0.58 mm in diameter in an electric circuit with a period of 40 μs, a capacitance of 4.2 $\mu\Phi$, and an initial voltage of 30 kV undergoes a complete stratification over a time of 200 ns, which is shorter than the acoustic time scale. The average distance between the striations was 0.78 mm in the experiments, at least twice the size of the length scale of the structure at the beginning of the explosion. According to our model, this dimension is $l = 0.336$ mm. A lower estimate of l at the end of the explosion yields $l_c \geqslant 2l = 0.672$ mm, which is in fair agreement with the experiments of Ref. 2.

In summary, we have established an analogy between the initial stages of the nucleation of turbulence in an incompressible viscous liquid and the electrical explosion of a conductor.

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