Exact solutions for the field moment functions in a randomly inhomogeneous medium. Parabolic equation

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Exact solutions are derived from a parabolic equation for the moment functions of a wave field in an arbitrary randomly inhomogeneous medium.

A scattering of waves of various types by media with random characteristics underlies several physical effects. The attempts which have been undertaken over the course of many years to solve the problem of wave scattering by randomly inhomogeneous media have resulted in the development of numerous approximate approaches. Exact results, in contrast, have been derived by appealing to unrealistic models of the fluctuating parameters of the nature of Markovian random functions.[1,2]

In principle, purely mathematical difficulties are concentrated at the heart of the problem. By working from dynamic equations for the wave field one can without difficulty construct closed equations with variational derivatives for certain probability characteristics. However, essentially no methods have been developed for solving such equations. The record reveals cases in which the solutions of equations have simply been guessed, and one does not rule out the possibility of using a "method" of that sort for problems of this type.

For simplicity we start from a parabolic equation for the complex field amplitude[1] $v(\vec{\rho},x)$:

$$\frac{\partial v(\vec{\rho},x)}{\partial x}=\frac{i}{2k}[\Delta_\perp+k^2\epsilon(\vec{\rho},x)]v(\vec{\rho},x) \tag{1}$$

with the boundary condition $v(\vec{\rho},x)|_{x=0}=v(\vec{\rho},0)$. Here $\epsilon(\vec{\rho},x)$ may be either a real or a complex random function of the coordinates.

A consequence of Eq. (1) is the well-known closed system of equations for the probability characteristic:

$$\Phi_{n,m}[\omega^*(\vec{r}\,');\omega(\vec{r}\,');\vec{\rho}_1,x_1,\ldots,\vec{\rho}_{n+m},x_{n+m}]$$

$$=<\exp\left\{\frac{i}{2}\int\limits_{-\infty}^{\infty}d^3r'[\epsilon(\vec{r}\,')\omega^*(\vec{r}\,')+\epsilon^*(\vec{r}\,')\omega(\vec{r}\,')]\right\}$$

$$\cdot v(\vec{\rho}_1,x_1)\ldots v(\vec{\rho}_n,x_n)v^*(\vec{\rho}_{n+1},x_{n+1})\ldots v^*(\vec{\rho}_{n+m},x_{n+m})>$$

Here $[\vec{r}\,'=(\vec{\rho}\,',x')]$. This system of equations is

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$$\frac{\partial \Phi_{n,m}}{\partial x_j} = \frac{i}{2k}\Delta_{\perp j}\Phi_{n,m} + k\frac{\delta \Phi_{n,m}}{\delta \omega^*(\vec{\rho}_j, x_j)}, \quad (j = 1, \ldots, n),$$

$$\frac{\partial \Phi_{n,m}}{\partial x_j} = -\frac{i}{2k}\Delta_{\perp j}\Phi_{n,m} - k\frac{\delta \Phi_{n,m}}{\delta \omega(\vec{\rho}_j, x_j)}, \quad (j = n+1, \ldots, n+m). \tag{2}$$

The customary notation is being used in (2), with the variational differentiation being carried out in the sense of a formal independence of ω and ω^*.

At $x_1 = x_2 = \ldots = x_{n+m} = 0$, i.e., at the boundary, we assume that $\Phi_{n,m}$ is known. Let us assume for simplicity that the boundary field $v(\vec{\rho},0)$ is statistically independent of the field $\epsilon(\vec{\rho},x)$. We can then write

$$\Phi_{n,m}|_{x_1=0,\ldots,x_{n+m}=0} = \theta^\epsilon[\omega^*(\vec{r}\ ');\omega(\vec{r}\ ')]\cdot$$

$$\cdot < v(\vec{\rho}_1, 0)\ldots v(\vec{\rho}_n, 0) v^*(\vec{\rho}_{n+1}, 0)\ldots v^*(\vec{\rho}_{n+m}, 0) >, \tag{3}$$

where θ^ϵ is a characteristic functional of the field $\epsilon(\vec{\rho},x)$.

Direct substitution verifies that

$$\Phi_{n,m}[\omega^*(\vec{r}\ ');\omega(\vec{r}\ ');\vec{\rho}_1, x_1, \ldots, \vec{\rho}_{n+m}, x_{n+m}]$$

$$= \exp\left[\frac{i}{2k}\left(\sum_{j=1}^{n} x_j\Delta_{\perp j} - \sum_{j=n+1}^{n+m} x_j\Delta_{\perp j}\right)\right]$$

$$\cdot\hat{T}\theta^\epsilon\left\{\omega^*(\vec{\rho}', x') + k\sum_{j=1}^{n}\int_0^{x_j} d\varsigma\,\delta(\varsigma - x')\exp[-i\frac{\varsigma}{2k}\Delta_{\perp j}]\delta(\vec{\rho}_j - \vec{\rho}')\exp[i\frac{\varsigma}{2k}\Delta_{\perp j}];\right.$$

$$\left.\omega(\vec{\rho}\ ', x') - k\sum_{j=n+1}^{n+m}\int_0^{x_j} d\varsigma\,\delta(\varsigma - x')\exp[i\frac{\varsigma}{2k}\Delta_{\perp j}]\delta(\vec{\rho}_j - \vec{\rho}\ ')\exp[-i\frac{\varsigma}{2k}\Delta_{\perp j}]\right\}$$

$$\cdot < v(\vec{\rho}_1, 0)\ldots v(\vec{\rho}_n, 0) v^*(\vec{\rho}_{n+1}, 0)\ldots v^*(\vec{\rho}_{n+m}, 0) >, \tag{4}$$

is a solution of Eqs. (2) under boundary condition (3). We are assuming that all the operators involved here act on all functions to their right. The operator $\hat{T}$ is a "chronological" operator,[3] representing a product, ordered along the longitudinal coordinate x, of the operators which appear in the integrands in the expansion of the characteristic functional in an operator series:

$$\hat{T}[\hat{L}(\varsigma_1)\ldots\hat{L}(\varsigma_m)] = \hat{L}(\varsigma_{\alpha 1})\hat{L}(\varsigma_{\alpha 2})\ldots\hat{L}(\varsigma_{\alpha m})$$

$$\varsigma_{\alpha 1} \geq \varsigma_{\alpha 2} \geq \ldots \geq \varsigma_{\alpha m}.$$

This chronological operator has to be introduced because the operators in the integrands with different values of the variable ζ do not commute.

Expression (4) is a complete solution of our problem, since the right side contains exclusively known functions and operators. Any moment functions of the random fields $v(\vec{\rho},x)$ and $\epsilon(\vec{\rho},x)$ can be found from (4) by the standard methods. For example, arbitrary moment functions of a random field $v(\vec{\rho},x)$ at $x>0$ can be found easily from (4) by setting ω^* and ω equal to zero. We wish to stress that random field $v(\vec{\rho},x)$ satisfies dynamic equation (1). There are no explicit limitations on the field $\epsilon(\vec{\rho},x)$. The assumption that the boundary field $v(\vec{\rho},x)$ and $\epsilon(\vec{\rho},x)$ are statistically independent is made for clarity; the generalizations are obvious.

The expressions for the moment functions may simplify substantially in specific cases. For example, in the case of a uniform, real Gaussian field $\epsilon(\vec{\rho},x)$ with a zero mean and a correlation function $B(\vec{r}_1-\vec{r}_2)$, we find the following expression for the mean field:

$$< v(\vec{\rho},x) >= \int_{-\infty}^{+\infty} d^2q\, f(\vec{q}) \exp\left\{ i(\vec{q}\vec{\rho} - \frac{x}{2k}|\vec{q}|^2) \right.$$

$$\left. -\frac{k^2}{4}\int_0^x d\varsigma(x-\varsigma) \int_{-\infty}^{+\infty} d^2\kappa F(\vec{\kappa},\varsigma) \cos\left(\frac{\varsigma}{2k}|\vec{\kappa}|^2\right) \exp\left(i\frac{\varsigma}{k}\vec{q}\vec{\kappa}\right) \right\},$$

where $f(\vec{q})$ and $F(\vec{\kappa},\zeta)$ are two-dimensional transforms of the Fourier functions $\langle v(\vec{\rho},0)\rangle$ and $B(\vec{\rho},\zeta)$ in $\vec{\rho}$. Klyatskin[2] has derived a special case of this solution (as for a moment function of second order) under the very restrictive assumption of a δ-correlated field $\epsilon(\vec{\rho},x)$ along x.

We note in conclusion that solutions of the type in (4) can be derived for more complex problems, e.g., by (for example) working from a wave equation or a Helmholtz equation for realizations of the wave field.

[1]S. M. Rytov, Yu. A. Kravtsov, and B. I. Tatarskiĭ, *Introduction to Statistical Radiophysics. Part 2 Random Fields*, Nauka, Moscow, 1978.
[2]V. I. Klyatskin, *Stochastic Equations and Waves in Randomly Inhomogeneous Media*, Nauka, Moscow, 1980.
[3]N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Wiley-Interscience, New York, 1980.

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