Correction $O(\alpha_s^3)$ to $\sigma_{tot}(e^+e^- \to$ hadrons) in quantum chromodynamics

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The results of a complete conversion of the QCD correction $O(\alpha_s^3)$ to the cross section for the annihilation of an electron–positron pair into hadrons are reported. This correction is negative in the standard $\overline{MS}$ scheme. The implications of the results are briefly discussed. In particular, the corresponding correction to the width of the decay $\tau \to \nu_\tau +$ hadrons is calculated.

Two years ago we published a calculation of the correction on the order of α_s^3 to the total cross section for the annihilation of e^+e^- into hadrons.[1] The correction which was found turned out to be unexpectedly large. Since then, in the course of a detailed analysis of the source of the predominant contribution to the results of Ref. 1, we have found errors in the computer program[2] which was used in the calculations. That program was written in the language of the SCHOONSCHIP analytic-calculation system. Consequently, the results which were found in Ref. 1 and the conclusions reached there regarding the physics are, unfortunately, incorrect.

In the present letter we are reporting the basic results of a complete recalculation of the correction $O(\alpha_s^3)$ to $\sigma_{tot}(e^+e^- \to$ hadrons) by means of a corrected program[2] (these results were presented previously at a conference[3]). We also discuss the conclusions reached on the basis of these new results.

A direct calculation was carried out in the Euclidean region $Q^2 = -q^2 > 0$ for the function

$$D(Q^2) = Q^2 \int_0^\infty \frac{R(s)}{(s+Q^2)^2} ds. \tag{1}$$

In the order of perturbation theory in which we are interested, the incorporation of effects of the analytic continuation of $D(Q^2)$ into the physical region leads to the appearance in $R(s)$ of an additional term, proportional to π^2:

$$R(s) = D(s) - \sum_f Q_f^2 \pi^2 \frac{\beta_0^2}{3} (\alpha_s/\pi)^3, \tag{2}$$

where β_0 is the first coefficient of the QCD β function, for which a three-loop expression is known in $\overline{MS}$-like schemes:[4]

$$\frac{1}{\pi}\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s) = -\beta_0(\alpha_s/\pi)^2 - \beta_1(\alpha_s/\pi)^3 - \beta_2(\alpha_s/\pi)^4, \tag{3}$$

where $\beta_0 = (11 - 2/3f)/4$, $\beta_1 = (102 - 38/3f)/16$, and $\beta_2 = (2857/2 - 5033/18f + 325/54f^2)/64$.

When the correction $O(\alpha_s^3)$ is taken into account, the renormalized expression for $R(s)$ becomes

$$R(s) = 3\Sigma Q_f^2\{1 + \alpha_s/\pi + (a_1 - a_2\ln(s/\mu^2))(\alpha_s/\pi)^2$$

$$+(b_1 - b_2\ln(s/\mu^2) + b_3\ln^2(s/\mu^2))(\alpha_s/\pi)^3\} - (\Sigma Q_f)^2 c_1(\alpha_s/\pi)^3. \quad (4)$$

In the $\overline{MS}$ scheme, the three-loop coefficient $O(\alpha_s^2)$ is not large:[5] $a_1 = 1.986 - 0.115f$. To find the correction $O(\alpha_s^3)$, it was necessary to calculate the counterterms of more than 100 four-loop diagrams which contribute to the photon propagator in QCD. The calculation method is described in detail in Refs. 5. The calculations were carried out with the help of the computer program (MINCER)[2] which we mentioned earlier, after we corrected it. This program implements an algorithm of integration by parts in a dimensional regularization.[6] Some of the diagrams which contribute to the photon propagator in QED and, correspondingly, to the four-loop β function of QED were recalculated on two different computers.[7] First, we used the original MINCER program,[2] implemented in the original version of the SCHOONSCHIP analytic-calculation program,[8] on a CDC-6500 computer at the Joint Institute for Nuclear Research. Second, we used a modification[9] of the program of Ref. 2 for an IBM version of SCHOONSCHIP, which was implemented on an ES-1037 computer at the Institute of Nuclear Research. The purely chromodynamic diagrams were recalculated by the program of Ref. 2.

Here are the numerical values of the coefficients in (4) which we used in the $\overline{MS}$ scheme:

$$b_1 = -6.637 - 1.200f - 0.005f^2, \quad c_1 = 1.239. \quad (5)$$

Here are the coefficients of the logarithmic terms: $a_2 = 2.75 - 0.167f$, $b_2 = 17.298 - 2.086f + 0.038f^2$, and $b_3 = 7.562 - 0.917f + 0.028f^2$. The latter coefficients can also be found from renormalization-group relations: $a_2 = \beta_0$, $b_2 = \beta_1 + 2\beta_0 a_1$, $b_3 = \beta_0^2$.

The coefficient of the α_s^3 terms is thus not very large numerically. Interestingly, most of this coefficient is drawn from the negative term which arises from the analytic continuation out of the Euclidean region into the region of physical energies.

Applying a renormalization group to (4) is equivalent to setting the logarithmic terms equal to zero and replacing $\alpha_s(\mu^2)$ by $\alpha_s(s)$. A running coupling constant can be expressed in terms of the logarithm $L = \ln(s/\Lambda^2_{\overline{MS}})$ in the following way:

$$\frac{\alpha_s}{\pi} = \frac{1}{\beta_0 L} - \frac{\beta_1 \ln L}{\beta_0^3 L^2} + \frac{1}{\beta_0^5 L^3}(\beta_1^2\ln^2 L - \beta_1^2\ln L + \beta_2\beta_0 - \beta_1^2). \quad (6)$$

To analyze the effect of the calculated correction on the determination of the parameter Λ through a fit of experimental data, we write the ratio $R(s)$ for the case of five active quark flavors:

$$R(s) = \frac{11}{3}\left(1 + \frac{\alpha_s}{\pi} + 1.409\left(\frac{\alpha_s}{\pi}\right)^2 - 12.805\left(\frac{\alpha_s}{\pi}\right)^3\right). \tag{7}$$

In Ref. 10, a fit of experimental data from e^+e^- colliders over the energy range 7–57 GeV yielded the following value for the QCD contribution to the cross section for e^+e^- annihilation: $r = (3/11)R(34\ \text{GeV}) \simeq 1.056 \pm 0.058$. At $O(\alpha_s^2)$, this value leads to $\alpha_s(34\ \text{GeV}) \approx 0.158 \pm 0.020$, which corresponds to $\Lambda_{\overline{MS}} \approx 440^{+300}_{-230}$ MeV (Ref. 10). The results of Ref. 10 agree with the results of a corresponding fit within the errors involved.[11]

We know quite well that terms on the order of α_s^2 are absolutely necessary for a correct determination of Λ, since they are the terms which fix the renormalization scheme. Consequently, the term on the order of α_s^3 which we calculated can be used to evaluate the theoretical uncertainty in the value of Λ determined from the fit. Incorporating the negative correction $O(\alpha_s^3)$ and using expression (6) for α_s, we find increased values of the constant and the parameter: $\alpha_s(34\ \text{GeV}) \approx 0.170 \pm 0.025$ and $\Lambda_{\overline{MS}} \approx 570^{+450}_{-320}$. Incorporating our correction increases the value of Λ by about 30%, and this figure is the theoretical uncertainty in Λ in the case at hand. The error interval also increases. The experimental errors in the determination of the parameter Λ thus presently outweigh the theoretical uncertainties in QCD contributions at high energies.

Let us examine the behavior of the perturbation-theory series in (7). With the values $\alpha_s \approx 0.17$ and $f = 5$ we find

$$r(34\,\text{GeV}) - 1 \approx 0{,}054 + 0{,}004 - 0{,}002. \tag{8}$$

It seems quite likely that the perturbation-theory series in QCD are asymptotic series. One might hope that the error of a truncated QCD series would be determined by the last term retained. The error of the right side of (8) (which contains only the contributions of terms containing α_s) could then be estimated to be $\approx 4\%$. However, one should bear in mind the question of the signs of the subsequent terms of this perturbation-theory series.

We turn now to the case of low energies (i.e., the case of large values of α_s). In this region, the e^+e^- collider data can be analyzed with the help of either the final-energy sum rules,[12]

$$R_k = \int_0^{s_0} R(s)s^k ds = \frac{s_0^{k+1}}{k+1}\{1 + (\alpha_s/\pi) + (\alpha_s/\pi)^2(a_1 + a_2/(k+1))$$

$$+(\alpha_s/\pi)^3(b_1 + b_2/(k+1) + 2b_3/(k+1)^2)\},\quad \alpha_s = \alpha_s(s_0), \tag{9}$$

or the Borel sum rules,[13]

$$M_n = \frac{1}{M^2}\int_0^{\infty} R(s)e^{-s/M^2}(s/M^2)^n ds. \tag{10}$$

In particular, with $f=3$ the perturbation-theory contributions to the first two Borel sum rules are

$$M_0 - 1 = \alpha_s/\pi + (\alpha_s/\pi)^2 2.939 + (\alpha_s/\pi)^3 6.297, \tag{11}$$

$$M_1 - 1 = \alpha_s/\pi + (\alpha_s/\pi)^2 0.690 - (\alpha_s/\pi)^3 10.924, \tag{12}$$

where $\alpha_s = \alpha_s(M^2)$. The term on the order of α_s^3 in series (12) becomes comparable to the preceding term at $\alpha_s \approx 0.2$. For the value $\alpha_s \approx 0.4$, which was found in the analysis of Refs. 14 and 15 (among other places), we thus have the problem of dealing with the calculated correction. If we are interested in the value of $R(s)$ itself, a similar situation arises for $f=3$ at $\alpha_s \approx 0.5$, for $f=4$ at $\alpha_s \approx 0.4$, and for $f=5$ at $\alpha_s \approx 0.35$. If the asymptotic regime of this series is assumed to begin with the corrections on the order of α_s^3 in these cases, then the incorporation of these corrections may degrade the accuracy of the resulting approximations and the values of the parameter Λ.

Let us now use this correction to calculate the corresponding approximation of the semihadronic decay width of a τ lepton or, more precisely, the quantity $R_\tau = \Gamma(\tau^- \to \nu_\tau + \text{hadrons})/\Gamma(\tau^- \to \nu_\tau' e^- \bar{\nu}_e)$. We know that this quantity can be used to find α_s in the low-energy region,[16,17] since the nonperturbative QCD corrections to it are small. The expression for R_τ is

$$R_\tau = 2 \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} (1 - s/M_\tau^2)^2 (1 + 2s/M_\tau^2) \tilde{R}(s), \tag{13}$$

where M_τ is the mass of the τ lepton, and $\tilde{R}(s)$ is the value of $R(s)$ found from (4) with $3\Sigma Q_f^2$ replaced by $3\Sigma|V_{ff'}|^2$, and with $(\Sigma Q_f)^2$ set equal to zero (here $V_{ff'}$ are the elements of the Kobayashi-Maskawa matrix, and the sum in the case of the τ lepton for the three flavors is $|V_{ud}|^2 + |V_{us}|^2 \simeq 1$). This expression is the final-energy sum rule. Using integration formula (10) and the results in (5), we find

$$R_\tau = 3(1 + \alpha_s/\pi + 5.20(\alpha_s/\pi)^2 + 26.38(\alpha_s/\pi)^3), \tag{14}$$

where $\alpha_s = \alpha_s(M_\tau^2)$.

With the electroweak corrections taken into account,[18] this result could be used for a detailed comparison of theoretical predictions with experimental data, in particular, with possible future data from $c - \tau$ factories.

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