Dependence of the effective Vilkovisky–De Witt action in quantum gravitation on the metric in the space of fields

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The effective Vilkovisky–De Witt action in quantum gravitation is studied as a function of the metric of the configuration space. An explicit expression describing this dependence is given for an Einstein quantum gravitation on a plane $R_k\, T_{d-k}$ background.

The effective action in quantum field theory is known to depend on the particular parametrization of the fields (and on the particular choice of the gauge condition in the case of gauge theories). In an effort to solve such problems, Vilkovisky and Barvinsky[1] and De Witt[2] constructed an effective action which, in gauge theories, is parametrization-invariant, gauge-invariant, and independent of the choice of gauge, both off and on the mass shell. It appears that the Vilkovisky–De Witt formulation[3,4] solves certain long-standing problems in quantum field theory and quantum gravitation. It has been the subject of a large number of studies (e.g., Refs. 1–9) in quantum gravitation. Unfortunately, the formalism of a parametrization-and gauge-independent effective action has its own problems, which we discuss below.

We recall that the parametrization- and gauge-independent effective action is defined by[1–4]

$$\exp \frac{i}{\hbar}\hat{\Gamma}[v^i;\phi_*] = \int D\mu[\phi]\mathrm{Det}\theta^\alpha_\beta\delta(\chi^\alpha)\exp\frac{i}{\hbar}(S[\phi] + \frac{\delta\hat{\Gamma}[v^i;\phi_*]}{\delta v^i}(\sigma^i(\phi_*,\phi)-v^i), \quad (1)$$

where $\sigma^i(\phi_*,\phi)$ are the Gaussian normal coordinates of the point ϕ, with origin at ϕ_* (Refs. 1 and 2), $v^i \equiv \langle\sigma^i(\phi_*,\phi)\rangle$, θ^α_β is a ghost operator, and χ^α is a gauge condition (see Refs. 3 and 4 for the details).

First, it is easy to see that expression (1) determines an infinite, single-parameter family of gauge-invariant and gauge-independent effective actions. The quantity ϕ_* plays the role of a parameter here. How are we to choose ϕ_*? There are, of course, the generalized Ward identities,[2] which control the dependence on ϕ_*. Note, however, that the gauge dependence in the standard effective action is also controlled by corresponding Ward identities.

This dependence on ϕ_* is a serious problem in the formalism of this new effective action. A convenient representation of the family of effective actions in (1), and the representation which is most commonly used, is the so-called Vilkovisky–De Witt effective action,[2-4] in which ϕ_* is defined by

$$\Gamma_{VD}[\bar{\phi}] = \hat{\Gamma}[v=0; \phi_* = \bar{\phi}], \tag{2}$$

where $\sigma^i(\phi_*, \bar{\phi}) = v^i$. It is the Vilkovisky–De Witt effective action which has been used for applications in quantum gravitation.[5-9]

A basic element required for calculating the Vilkovisky–De Witt effective action is the metric γ_{mn} in the space of fields $g_m \equiv g_{\mu\nu}$. The most general form for this metric in quantum gravitation—a form which is compatible with definition[1,2] (1), (2)—is

$$\gamma_{mn} \equiv \gamma^{\mu\nu\rho'\sigma'}(x,x') = \sqrt{g(x)}\{\frac{1}{2}g^{\mu\rho'}(x)g^{\nu\sigma'}(x)$$

$$+\frac{1}{2}g^{\mu\sigma'}(x)g^{\nu\rho'}(x) - a g^{\mu\nu}(x)g^{\rho'\sigma'}(x)\}\delta(x,x'), \tag{3}$$

where a is a numerical parameter. If we require that metric (3) be the same as the matrix in front of the term with the higher derivative in the classical action, we could determine a. However, this is not a necessary condition. Furthermore, that requirement does not lead to a unique result. For example, in an Einstein gravitation it would lead to[1] $a = 1/2$, while in an R^2 gravitation the same requirement would lead to a different a.

It is worthwhile to study the explicit dependence of the Vilkovisky–De Witt effective action on the parameter a (i.e., on the metric on the space of fields). As an example we consider an Einstein gravitation with the action

$$S = -\frac{1}{\kappa^2}\int d^dx\sqrt{g}(R-\Lambda) \tag{4}$$

on a plane $R_k T_{d-k}$ background. An explicit calculation of Γ_{VD} in the single-loop approximation with metric (3) yields[1)]

$$\Gamma_{VD} = \frac{\Lambda}{\kappa^2}\Pi_{i=1}^{d-k}(2\pi\rho_i)\int d^kx + \frac{1}{2}\left\{\frac{d^2-d-2}{2}\mathrm{Sp}\,\ln(\Box + b\Lambda) - \right\}$$

$$\times\left\{-d\mathrm{Sp}\,\ln\Box + \mathrm{Sp}\,\ln\left[\Box + \frac{(da-1)b\Lambda}{(d-2)(1-a)}\right]\right\}, \tag{5}$$

where $b = d/4(da-1)$, and ρ_i are the radii of the torus. With $a = 1/2$, this expression becomes the same as that found in Refs 5 and 6, while with $d = 5$ it becomes the same as that found in Refs. 5 and 7. An explicit expression for $\mathrm{Sp}\,\ln(\Box + X)$ is given in Refs. 5 and 6, among other places.

The Vilkovisky–De Witt effective action thus depends explicitly on the metric γ_{mn} (on the parameter a). This dependence leads, in particular, to an a dependence of the spontaneous-compactification radii found through the use of the standard conditions.[5-7] We are thus confronted with a dilemma: a gauge dependence of the spontaneous-compactification radii, if we use the standard effective action, or an a dependence, if we use the Vilkovisky–De Witt effective action. One can, of course, fix a, as was done in Refs. 1–9. However, we could just as well fix the gauge in the standard effective action, choosing some "physical" gauge. In order to resolve this problem, it will be necessary to carry out a more detailed study of the Vilkovisky–De Witt effective action (which is not "unique," as is sometimes asserted in the literature).

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[1)] The details of these calculations will be published separately.

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