A linear extension of the Virasoro algebra

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A linear extension of the conformal group which is generated by auxiliary currents with spin 2 is constructed. The structure and representations of these algebras and their relation to the group $SU(\infty)$ are discussed.

Conformal field theory in two-dimensional space allows a unified descriptions of the behavior of dynamical systems at the critical point and the calculation of their correlation functions and the spectrum of anomalous dimensions.[1] The classification of local fields according to representations of the conformal group plays a fundamental role in the solution of the "bootstrap" equations, where, as is well known, the solutions possess a symmetry of higher degree than the conformal symmetry[2] and contain the Virasoro algebra as a subalgebra.

It is interesting to further extend the dynamical symmetries of local fields and generalize the theory to a space of higher dimension.

Let us consider the current algebra

$$\{J_i(x), J_k(y)\} = \omega_{ik}\partial_x\delta(x-y), \tag{1}$$

where $i,k = 1,...,$ ω_{ik} is a constant symmetric matrix. We construct a new system of currents which depends quadratically on the original current (1)

$$T^a(x) = \frac{1}{2}J_i M^a_{ik} J_k + \alpha_i M^a_{ik}\partial_x J_k, \tag{2}$$

where these currents form a closed algebra if the matrices M are symmetric:

$$\{T^a(x), T^b(y)\} = F^{ab}_d[\partial_x T^d + 2T^d\partial_x + C^d\partial^3_x]\delta(x-y), \tag{3}$$

and the structure constants F satisfy the system of equations

$$M^a_{ij}\omega_{jk}M^b_{kl} = F^{ab}_d M^d_{il}. \tag{4}$$

The classical central charge in (3) is

$$C^d_{\rm cl} = \alpha_i M^d_{ik}\alpha_k. \tag{5}$$

Here the number of currents T may be unequal to the number of currents J. The algebra (3) of particular interest to us was introduced in Refs. 3 and 4.

We assume that the Kac–Moody algebra of the currents (1) is specified by an arbitrary, nondegenerate, symmetric matrix ω_{ik}, and it is necessary to find a set of matrices M satisfying equation (4) together with the structure constants F. Naturally, algebras which are not isomorphic to each other are also of interest, so we shall consider all possible linear transformations of the currents J and T. We can diagonalize the matrix ω by means of unitary transformations of the currents J. The transformed matrices M will, as before, be symmetric and satisfy the equation

$$M^a M^b = F_c^{ab} M^c, \tag{6}$$

from which we see that they commute with each other $[M^a, M^b] = 0$ and form a commutative ring.

As far as the structure constants F are concerned, they are invariant under these unitary transformations.

Let us now consider a linear transformation of the currents $T(T^a = \Omega_b^a T^b)$, in which the constants F now transform as tensors and the matrices M transform as vectors. We can study several different solutions of (6) which are not isomorphic to each other.

1) If the matrix $\|\lambda\|$ constructed from the eigenvalues of $M(M^a \psi = \lambda^a \psi)$ is nondegenerate, $\det\|\lambda\| \neq 0$, then the current algebra (3) is equivalent to a direct sum of Virasoro algebras. Let us consider the basis matrices

$$e^i = \mathrm{diag}\,(0...1...0), \quad e^i e^j = \delta^{ij} e^j, \tag{7}$$

so that $M^a = \lambda_i^a e^i$. The latter equation can be interpreted as the transformation of the currents T from one basis to another with the matrix $\Omega = \|\lambda\|$. On the other hand, in the basis (7) the fundamental algebra (3) is equal to a direct sum of Virasoro algebras.

2) If the matrix $\|\lambda\|$ is degenerate, but there are no nilpotent matrices among the matrices M, i.e., matrices for which all the eigenvalues are equal to zero, the current algebra (3) is again a sum of Virasoro algebras, though, of course, a smaller number of them.

3) If among the matrices M there are nilpotent matrices, they cannot be expressed in terms of the basis matrices (7), and the current algebra (3) cannot be expressed as a direct sum of Virasoro algebras.

These three cases exhaust the nonisomorphic solutions of equation (6).

Expanding the currents J and T in a series, we obtain the following system of commutation relations from (1) and (3) for the coefficients T_n^i and L_n^a:

$$[L_n^a, L_m^b] = F_d^{ab}\left[(n-m)L_{n+m}^d + \frac{C^d}{12}(n^3 - n)\delta_{n+m,0}\right], \tag{8a}$$

$$[L_n^a, T_m^i] = -n M_{ij}^a T_{n+m}^j, \tag{8b}$$

$$[T_n^i, T_m^j] = n\delta^{ij}\delta_{n+m,0}, \tag{8c}$$

where $J_i(x) = T^i_n \exp(-inx) = T^i_n Z^{-n}$, $T^a(x) = L^a_n Z^{-n}$, and relation (2) takes the form

$$L^a_n = \frac{1}{2} : \quad T^i_{n-m} M^a_{ij} T^j_m : \tag{9}$$

In the latter expressions we have ignored terms proportional to α.

Expression (9) is the natural generalization of the Sugawara construction when M is different from unity. By standard calculations we can obtain the expression for the central charge[5]

$$C^d_{cl} = \mathrm{tr} M^d, \tag{10}$$

and if the α dependence is restored in (8), for the total charge we find

$$C^d_{tot} = \mathrm{tr} M^d - 24\alpha_0 M^d \alpha_0, \tag{11}$$

where we have used the normalization found in the literature $\alpha_i \to i\sqrt{2}\alpha^0_i$.

We are interested in the extension of the Virasoro algebra (3), (8a), because it can be used to construct representations of the algebra of $SU(\infty)$ which arises in $SU(N)$ gauge theories in the limit $N \to \infty$. In fact, the structure constants of the $SU(N)$ algebra in the two-index basis $J_{n_1 n_2}$ have the form $[J_{\bar{n}}, J_{\bar{m}}] = 2\pi/N \sin[(2\pi/N)\bar{n}\Lambda\bar{m}] J_{\bar{n}+\bar{m}}$ (Ref. 6) and for $N \to \infty$ coincide with the structure constants of the algebras of area-preserving diffeomorphisms of the torus T^2, $[L_{\bar{n}}, L_{\bar{m}}] = (\bar{n}\Lambda\bar{m}) L_{\bar{n}+\bar{m}}$. The latter algebra coincides with the subalgebra of arbitrary diffeomorphisms of the torus[8,9]

$$[L^1_n, L^1_m] = (n_1 - m_1) L^1_{n+m}, \qquad [L^2_n, L^2_m] = (n_2 - m_2) L^2_{n+m}$$

$$[L^1_n, L^2_m] = -m_1 L^2_{n+m} + n_2 L^1_{n+m} \tag{12}$$

if we set $L_{\bar{n}} = n_2 L^1_{\bar{n}} - n_1 L^2_{\bar{n}}$. We shall show that these algebras can be constructed using the basis algebra (8a).

Let the eigenvalues of the matrix M^a be equal to each other and proportional to the Nth root of unity $M^a = \omega^a I$, $\omega^N = 1$. Then $F^{ab}_c = \delta_{a+b,c}$ modulo N, so that

$$L_{na} \equiv L^a_n = \frac{1}{2}\omega^a : \; T^i_{n-m} T^i_m :, \tag{13}$$

and the algebra (8a) takes the form

$$[L_{na}, \; L_{mb}] = (n-m) L_{n+m,a+b} + \frac{C_{a+b}}{12}(n^3 - n)\delta_{n+m,0}\,. \tag{14}$$

For $N \to \infty$ it reproduces the subalgebra in (12).

Let us now consider the third case, where the matrices M are nilpotent. In this case the basis algebra (8a), (3) is not isomorphic to the direct sum of Virasoro algebras. Using the Jordan normal form of the matrix M, it can be proved that the nilpotent matrices M form a commutative ring when and only when they have the form

$$M^a = (1, e, e^2, \dots e^{N-1}),$$
$$e = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \qquad (15)$$

and $e^N = e^{N+1} = \dots = 0$. The matrix M^0 corresponds to the usual energy-momentum tensor $T(Z)$ of the conformal theory, and the other matrices correspond to new fields $T^a = Q^a_n Z^{-n}$ with conformal spin $S = 2$. They determine the algebra

$$[L_n, \ Q^a_m] = (n-m)Q^a_{n+m} + \frac{C_a}{12}(n^3 - n)\delta_{n+m,o},$$
$$[Q^a_n, \ Q^b_m] =$$
$$= \begin{cases} (n-m)Q^{a+b}_{n+m} + \frac{C_{a+b}}{12}(n^3 - n)\delta_{n+m,0}, & \text{if} \quad 1 \le a+b \le N-1; \\ 0, & \text{if} \quad a+b \ge N, \end{cases} \qquad (16)$$

which is no longer isomorphic to the direct sum of Virasoro algebras. We note that the simplest of the minimal models containing a primary field of spin 2 is $M(14/15)$.

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