Probability distribution of intensity fluctuations of electromagnetic wave in region of strong fluctuations

R. Kh. Almaev and A. A. Suvorov
Taĭfun Scientific-Industrial Association

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An expression is derived for the probability distribution of strong fluctuations in the intensity I of radiation in a turbulent medium. This expression gives a satisfactory description of experimental data.

The propagation of electromagnetic waves through media of various types (a turbulent atmosphere, the ionosphere, the interstellar medium, etc.), in which there are fluctuations in the dielectric constant ϵ, is accompanied by accumulating random changes in the phase and amplitude of the wave. The magnitude and distribution of these changes are determined by the properties of the medium and the path length. The basic characteristics of a fluctuating radiation field in a turbulent medium have now been studied quite well. This comment applies in particular to the region of strong intensity fluctuations, which is characterized by a saturation of the relative variance of the fluctuations in I. So far, on the other hand, we do not have a satisfactory explanation for the discrepancy between the behavior of the theoretical and experimental probability distributions for intensity fluctuations. This is the "log-normal paradox."[1] Experimental results describe a $P(I)$ distribution which is approximately a logarithmically normal distribution,[1,2] while the theory predicts that the probability distribution asymptotically approaches a Rayleigh distribution in a region of strong fluctuations.[1,3] Attempts to resolve this paradox by approximating $P(I)$ by a piecewise function[4] or a K distribution[5] or by constructing a heuristic model for $P(I)$ (Ref. 6) have ultimately failed to provide the expected results, since all these distributions asymptotically ap-

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proach a Rayleigh distribution with increasing path length z, and a Rayleigh distribution is at odds with the experimental data.

In the present study we have derived a function $P(I)$ which agrees satisfactorily with the experimental data. We have used the method of calculating the nth moment of the radiation intensity,[3] modified for the case in which a wave is propagating through a turbulent medium with fluctuations in both the real and imaginary components of ϵ. The results show that the improvement in the agreement between theory and experiment stems from the incorporation of a random weakening of the wave by fluctuations in the imaginary part of ϵ.

Let us consider the propagation of an electromagnetic wave through a randomly inhomogeneous medium with a complex ϵ field with a minimum length scale l_0. The fluctuations in it are much greater than the radiation wavelength $\lambda\,(l_0 \gg \lambda)$. In this case the complex amplitude $U(\vec{\rho},z)$ of monochromatic radiation satisfies the parabolic equation (Ref. 7, for example)

$$2ik\frac{\partial}{\partial z}U + \Delta_{\perp}U + k^2[\tilde{\epsilon}(\vec{\rho}, z) + i\bar{\epsilon}_I]U = 0, \tag{1}$$

where $\tilde{\epsilon} = \epsilon - \epsilon_0/\epsilon_0 = \tilde{\epsilon}_R + i\tilde{\epsilon}_I$ are the relativistic fluctuations in ϵ; $\tilde{\epsilon}_R$ and $\tilde{\epsilon}_I$ are the real and imaginary parts of $\tilde{\epsilon}$; $\epsilon_0 = \mathrm{Re}\langle\epsilon\rangle$ and $\bar{\epsilon}_I = \mathrm{Im}\,\langle\epsilon\rangle/\epsilon_0$ are the average values of the real and imaginary parts of ϵ, which we assume for simplicity to be constants; z is the coordinate along the axis which coincides with the original wave propagation direction; $\vec{\rho} = \{x,y\}$ and $\Delta_{\perp}$ are the radius vector and Laplacian in the $z=$ const. plane; and $k = (2\pi/\lambda)\sqrt{\epsilon_0}$. The boundary value of the complex amplitude, $U(\vec{\rho},z)|_{z=0} = U_0(\vec{\rho})$, and the field of fluctuations in ϵ are assumed to be given. In particular, for fluctuations in ϵ which are δ-correlated along z we can write

$$\langle \tilde{\epsilon}_q(\vec{\rho}_1, z_1)\tilde{\epsilon}_{q'}(\vec{\rho}_2, z_2)\rangle = \delta(z_1 - z_2)A_{qq'}(\vec{\rho}_1\vec{\rho}_2), \tag{2}$$

$$A_{qq'}(\vec{\rho}) = 2\pi\iint d^2\kappa\,\Phi_{qq'}(\vec{\kappa}, 0)\cos\vec{\kappa}\vec{\rho}, \tag{3}$$

where $\Phi_{qq'}(\vec{\kappa},\kappa_z)$ is the 3D spectrum of the components of $\tilde{\epsilon}$. For media with a well-developed turbulence, this spectrum is a Kolmogorov power-law spectrum $\Phi_{qq'}(\vec{\kappa}) \sim C^2_{qq'}\kappa^{-11/3}$, where $C^2_{qq'}$ is a structural characteristic of the real $(q = q' = R)$ and imaginary $(q = q' = I)$ components of $\tilde{\epsilon}$ and of their correlations $(q = I,\, q' = R)$. The angle brackets mean an average over the ensemble of realizations of $\tilde{\epsilon}$.

The asymptotic behavior of the nth moment of the intensity, $\langle I^n\rangle = \langle (C/4\pi|U|^2)^n\rangle$, in a region of strong fluctuations $(\beta_0^2(z) = 0.31C^2_{RR}k^{7/6}z^{11/6} \gg 1)$ is found by writing the solution of Eq. (1) in terms of a Green's function in the form of a Feynman path integral.[8] Under the assumption that the correlations in $\tilde{\epsilon}_R$ and $\tilde{\epsilon}_I$ contribute negligibly to the fluctuations in I—an assumption which is valid under the condition $(C^2_{RI}/C^2_{RR})\beta_0^{3/5}(z) \ll k^2A_{II}(0)z$—and following Ref. 3, we can write the following expression for $\langle I^n\rangle$ in a region of strong fluctuations:

$$\langle I^n(\vec{\rho}, z)\rangle \approx n! \exp\left\{ \frac{n^2}{2} k^2 A_{II}(0) z \right\} \hat{L} e^{\Psi(\{\mathbf{V}_\alpha\}_{\alpha=1,\dots 2n})}, \tag{4}$$

where

$$\hat{L} = |C|^{2n} \prod_{j=1}^{n} \iint d^2\rho'_{2j-1} d^2\rho'_{2j} \Gamma_{20}(\vec{\rho}'_{2j-1}, \vec{\rho}'_{2j}) G_0(\vec{\rho}, z; \vec{\rho}'_{2j-1}, 0) G_0^*(\vec{\rho}, z; \vec{\rho}'_{2j}, 0)$$

$$\times \iint D^2 V_{2j-1}(\xi) D^2 V_{2j}(\xi) \exp\left\{ \frac{ik}{2} \int_0^z d\xi [\dot{V}^2_{2j-1}(\xi) - \dot{V}^2_{2j}(\xi)] - \frac{k^2}{4} \int_0^z d\xi D_{RR}(\mathbf{R}_{2j-1}(\xi)\right.$$

$$\left. - \mathbf{R}_{2j}(\xi) + \mathbf{V}_{2j-1}(\xi) - \mathbf{V}_{2j}(\xi)) \right\}; \quad \mathbf{R}_\alpha(\xi) = \vec{\rho} \frac{\xi}{z} + \vec{\rho}'_\alpha \frac{(z-\xi)}{z}; \quad \Psi(\{\mathbf{V}_\alpha\}_{\alpha=1,\dots 2n})$$

$$= \frac{k^2}{8} \sum_{l,m=1}^{2n}{}' (-1)^{l+m} \int_0^z d\xi D_{RR}(\mathbf{R}_l(\xi) - \mathbf{R}_m(\xi) + \mathbf{V}_l(\xi) - \mathbf{V}_m(\xi)),$$

$$D_{qq'}(\vec{\rho}) = A_{qq'}(0) - A_{qq'}(\vec{\rho}),$$

$$\Gamma_{20}(\vec{\rho}'_\alpha, \vec{\rho}'_{\alpha+1}) = \frac{c}{4\pi} U_0(\vec{\rho}'_\alpha) U_0^*(\vec{\rho}'_{\alpha+1}); \quad \text{here} \quad G_0(\vec{\rho}, z; \vec{\rho}', 0) = \frac{k}{2\pi i z}$$

$$\times \exp\left\{ \frac{ik}{2z} |\vec{\rho} - \vec{\rho}'|^2 + ikz - \frac{k}{2} \bar{\epsilon}_I z \right\}$$

is the Green's function of a medium without fluctuations in ϵ. The prime on the sums over l and m means that the sums do not include the functions D_{RR} in $\hat{L}$. Here C is a normalization constant, and $\dot{V}(\xi) = (d/d\xi) V(\xi)$.

Using a variational principle, we can describe the effect of the operator $\hat{L}$ on e^Ψ in the region $\beta_0^2 \gg 1$ by means of the relation

$$\hat{L} e^\Psi \approx \Phi_0 e^{\Phi_1}, \tag{5}$$

where $\Phi_N = (1/N!) \hat{L} \Psi^N \sim (\beta_0^{-4/5})^N$.

Using (5), we finally find the following expression for the nth moment of the intensity of a plane wave, I_p, and a spherical wave, I_s:

$$\langle I_\alpha^n \rangle \approx n! \langle I_\alpha \rangle^n \exp\left\{ \frac{n(n-1)}{2} \Omega_\alpha^2 \right\}. \tag{6}$$

Here $\Omega_\alpha^2 = k^2 A_{II}(0) z + \frac{1}{2}\gamma_\alpha \beta_0^{-4/5}(z)$, $\alpha = p,s$; $\langle I_\alpha \rangle = I_0 b_\alpha \exp\{-k\tilde{\epsilon}_I z + \frac{1}{2} k^2 \cdot A_{II}(0) z\}$; $b_p = 1$; $b_s = (kz)^{-2}$; $\gamma_p = 0.86$; and $\gamma_s = 2.8$. The probability distribution of the radiation intensity corresponding to (6) is

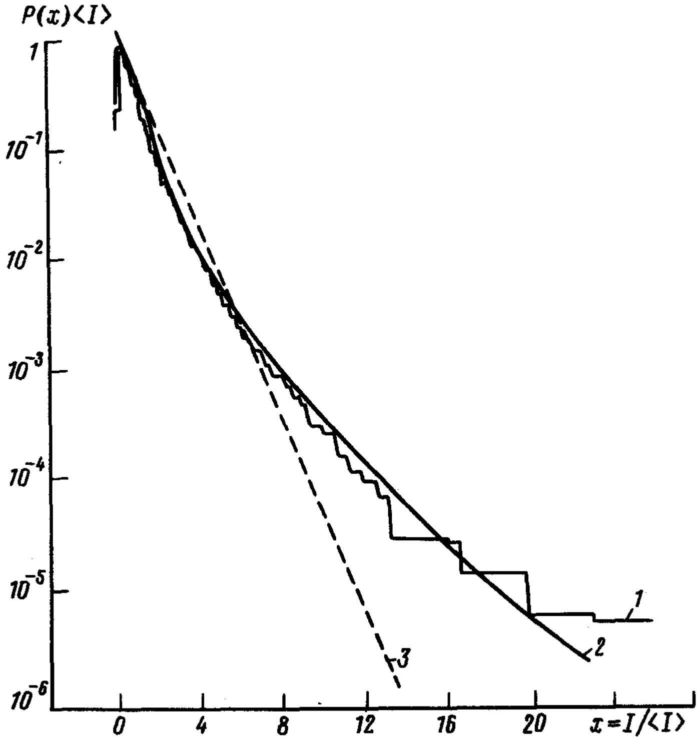

FIG. 1. Probability distribution of the relative intensity fluctuations. 1—Experimental histogram from Ref. 2; 2—distribution (7) ($\Omega_p^2 = 0.21$); 3—Rayleigh distribution.

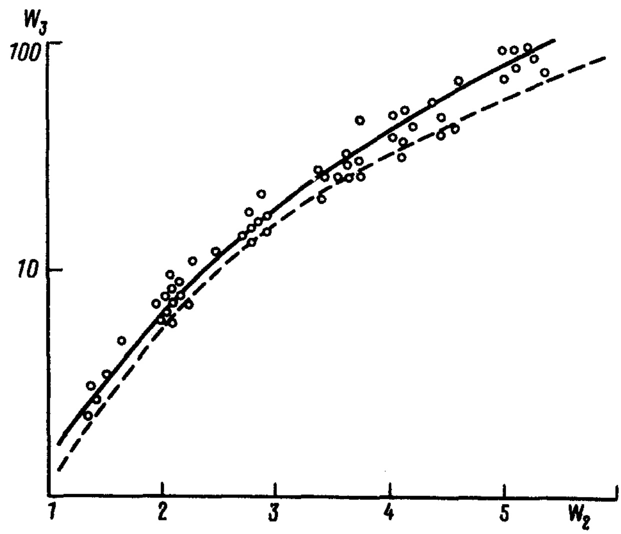

FIG. 2. The third moment of the intensity as a function of the second. ○—Experimental data of Ref. 5; solid line—calculated from the formula $W_3 = 3!(W_2/2)^3$; dashed line—K distribution.[5]

$$P_\alpha(I) = \frac{1}{\sqrt{2\pi}\Omega_\alpha \langle I_\alpha\rangle} e^{\Omega_\alpha^2/2} \int d\tau \exp\{-\frac{\tau^2}{2\Omega_\alpha^2} + \tau - \frac{I}{\langle I_\alpha\rangle} \exp(\frac{\Omega_\alpha^2}{2} + \tau)\}. \tag{7}$$

A characteristic feature of this distribution is that in the limit $\beta \to \infty$ it tends toward a logarithmically normal distribution; i.e., it corresponds to the distribution found experimentally. An expression for $P(I)$ similar to (7) was derived in Ref. 6 on the basis of a heuristic model. Chunside and Clifford[6] did not consider the fluctuations $\tilde{\epsilon}_I$, however, and in the limit $\beta \to \infty$ (with $\Omega_\alpha \to 0$) the distribution $P(I)$ tended toward a Rayleigh distribution. In (7), the quantity Ω_α increases with increasing β_0.

Let us compare function (7) with the $P(I)$ distribution found experimentally[2] (Fig. 1). It can be seen from Fig. 1 that a satisfactory agreement between the theoretical and experimental distributions is achieved with $\Omega_p^2 = 0.21$. Under the experimental conditions ($z = 1.75 \times 10^5$ cm, $\beta_0^2 = 25$, $\sigma_I^2 = 1.46$), this value of Ω_p^2 corresponds to the value $A_H(0) = 5.2 \times 10^{-17}$ cm, which is characteristic of the atmosphere. Figure 2 shows plots of $W_3(W_2)$, which are normalized moments, $W_n = \langle I^n\rangle / \langle I\rangle^n$, which are coupled by the relation $W_n = n!(W_2/2)^{n(n-1)/2}$. Here again we see a satisfactory agreement between the theoretical and experimental data (the latter from Ref. 5).

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