Diffusion in random potential: Renormalization-group analysis

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The transport in an inhomogeneous medium with a dimensionality which is not too high results from a motion of a localization center. The laws of motion of this center and the conditions for a transition to ordinary diffusion are found as functions of the type of inhomogeneity and the dimensionality of the space.

Considerable progress has recently been achieved in the effort to reach an understanding of the properties of various manifolds (domain walls, vortex lines in superconductors, interfaces, etc.) in a random medium (see, for example, the review articles in Refs. 1 and 2). These systems are described by a Hamiltonian with the general form

$$H = \int d^D x \left[\frac{m}{2} (\nabla \mathbf{z})^2 + V(x, \mathbf{z}(x)) \right], \tag{1}$$

where m is the effective stiffness of the D-dimensional manifold, $\mathbf{z}$ is a d-dimensional vector representing the transverse displacement of this manifold, and v is a random potential with a zero mean. For an Abrikosov vortex, for example, we would have $D=1$, $d=2$, while for a domain wall in a 3D crystal we would have $D=2$, $d=1$. Cases in which the correlation function of the random potential falls off rapidly in the longitudinal (x) direction, $\langle V(x,\mathbf{z}) V(x',\mathbf{z}') \rangle = \Delta \delta(x-x') R_0(\mathbf{z}-\mathbf{z}')$, have been studied thoroughly. In the present letter we take up the degenerate case, in which the random potential has no explicit dependence on the longitudinal coordinates. At first glance, this model of a random medium appears rather contrived—something which could not be realized in practice. That perception is correct, provided that one is thinking of only the entities listed above. There are, on the other hand, a fairly large number of other physical systems in which a random potential in the form which we have selected is more the rule than the exception. Specifically, with $D=1$ the partition function $W(t,\mathbf{z})$ of a line which is pinned at the points (0,0) and (t,$\mathbf{z}$) obeys a diffusion equation of the nature of a d-dimensional Schrödinger equation:[3]

$$T \frac{\partial W}{\partial t} = \frac{T^2}{2m} \frac{\partial^2 W}{\partial \mathbf{z}^2} - V(t, \mathbf{z}) W. \tag{2}$$

The role of Planck's constant is played by the temperature T; the timelike variable is the length of the line, t; and the "mass" of the particle corresponds to the stiffness m. If we ignore the physical meaning of these parameters, which is associated with Hamiltonian (1), we see that Eq. (2) itself describes a d-dimensional diffusion of a certain quantity W in a random potential. As examples we might also cite autocatalytic chemical reactions in an inhomogeneous medium, the propagation of migrating popula-

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tions, biological evolution, etc.[4] In such cases, W is a probability density of the corresponding quantity. From the standpoint of those problems, whose solution is the goal of the present letter, the most natural formulation is obviously that in which the properties of the medium in which the diffusion occurs remain constant over time: $V(t,\mathbf{z}) = V(\mathbf{z})$. The properties of Hamiltonian (1), with a random potential in a form which seems at first glance to be unrealistic, are thus intimately related to the behavior of another real physical system. It is preferable to study the Hamiltonian, since there are some graphic physical considerations (discussed below) which make it a fairly simple matter to derive many of the results. We will retain the names used previously for the parameters m and T, although they could actually be other quantities. In addition, the time t could have the meaning of a length in the context of Hamiltonian (1).

Our starting point is Hamiltonian (1) with an arbitrary D and with a correlation function $\langle V(\mathbf{z})V(\mathbf{z}')\rangle = \Delta R_0(\mathbf{z}-\mathbf{z}')$ of a general type: $R_0(\mathbf{z}) = z^{-\beta d}$ as $\mathbf{z}\to\infty$. We will also discuss the case of an uncorrelated potential. The nonlinear renormalization-group functional equation which corresponds to Hamiltonian (1) with $V(x,\mathbf{z}(x)) = V(\mathbf{z}(x))$ differs from those given in Refs. 5, 6, and 2 only in the linear component:

$$\frac{dR}{dl} = (4-4\zeta)R + \zeta zR'(z) + \frac{1}{2}R''^2(z) - R''(z)R''(0) + \frac{d-1}{2}\frac{R'(z)}{z}(R'(z) - R''(0)). \qquad (3)$$

Here l is the ordinary renormalization parameter, and ζ is the so-called roughness index, which corresponds to the gauge dimensionality of the field,[1,2] $z(x)$ (it also characterizes a diffusion). In writing Eq. (3) we also assumed that R depends on only the absolute value of z and that the renormalized temperature, which satisfies the equation

$$\frac{dT}{dl} = (2-D-2\zeta)T \qquad (4)$$

is unimportant in the renormalization-group sense. Since the further analysis of Eq. (3) largely reproduces the arguments presented in detail in Refs. 5, 6, and 2, we will write only the final results. In the limit of large z, a fixed function of Eq. (3) consists of two terms,

$$R^*(z) = Az^{4-4/\zeta} + Bz^{(4/\zeta)-4-d}\exp(-\zeta z^2/2\epsilon), \qquad (5)$$

where A and B are arbitrary constants, and $\epsilon = -R''(0)$. The asymptotic form of (5) is determined by the first, second, and fourth terms on the right side of Eq. (3). None of the other terms here, and none of the terms (which we have not written out) of higher order in the derivatives of R, change the behavior of the fixed-point function at large distances, since the latter tends toward zero as $z \to \infty$. The first term in (5) corresponds to long-range correlations, and the second to a random potential of finite range. Equating the exponents of the first term and of the bare correlation function $R_0 = z^{-\beta d}$, we find for this case

$$\zeta_{LR}(\beta, d) = \frac{4}{4 + \beta d} . \tag{6}$$

The "long-range" fixed point which we have found, (5), (6), is stable under the condition[6,2] $\beta < 1/2$. Nonvanishing values of both the coefficient A and coefficient B correspond to it. At $\beta \geqslant 1/2$, the behavior over large scales (over long times) is determined by only the second term in (5) (the first is unimportant in the renormalization-group sense[2,6]). Since β does not enter Eq. (3), the value of the index ζ_{SR} at $\beta \geqslant 1/2$ (and also for an uncorrelated potential) is independent of β and can be found from continuity considerations:[2,6]

$$\zeta_{SR}(d) = \zeta_{LR}(\beta = 1/2, d) = \frac{8}{8 + d} . \tag{7}$$

We will now show how these results can be reproduced from simple physical considerations which date back to Imry and Ma.[7] The index ζ forms as a consequence of a compromise between the increase in the elastic part of the energy of the deformed manifold [the first term in (1)] and the decrease in the random part [the second term in (1)]. Let us consider a manifold of finite size t with a characteristic transverse displacement $\langle z \rangle$. For a long-range potential with $R_0 = z^{-\beta d}$, the condition of this compromise is

$$m(\langle z \rangle / t)^2 \approx (\Delta \langle z \rangle^{-\beta d})^{1/2} .$$

Hence,

$$\langle z \rangle \approx \left(\frac{\Delta}{m^2}\right)^{\frac{1}{4+\beta d}} t^{\frac{4}{4+\beta d}} . \tag{8}$$

Comparing with the definition of $\langle z \rangle \sim t^{\zeta}$, we find Eq. (6). At $\beta \geqslant 1/2$, and also for an uncorrelated potential, we find the following expression by substituting the result of the renormalization-group analysis, $\beta = 1/2$, into (8):

$$\langle z \rangle \approx \left(\frac{\delta}{m^2}\right)^{\frac{2}{8+d}} t^{\frac{8}{8+d}} . \tag{9}$$

Here δ is proportional to Δ and depends on the bare correlation function R_0. Knowing the latter, we can easily reconstruct it by dimensionality considerations. The "zero-temperature" fixed points corresponding to diffusion laws (8) and (9) are stable if the expression in parentheses in (4) (with $D = 1$) is negative. In other words, an anomalous diffusion (with ζ different from 1/2) occurs if

$$\begin{aligned} \beta d &< 4, \quad \beta < 1/2, \\ d &< 8, \quad \beta \geq 1/2. \end{aligned} \tag{10}$$

The "zero-temperature nature" has a clear physical meaning. We mentioned above that the temperature in Eq. (2) has the meaning of Planck's constant in the Schrödinger equation. The "zero-temperature nature" thus implies a classical nature. In classical mechanics, however, the trajectory of a particle is well defined, so the solution of Eq. (2) is a localization center (a sharp peak in the probability density) which is moving in accordance with laws (8) and (9) along progressively more advantageous minima of the random potential. The latter conclusion was also reached (without proof) in Ref. 4. Interestingly, the variational calculation carried out in that paper, on the basis of Mott's ideas,[8] led to the result $\langle z \rangle \sim t/\ln t$ for an uncorrelated random potential, regardless of the dimensionality of the space. In a space of low dimensionality, that result is numerically close to (7).

If the dimensionality of the space does not satisfy inequalities (10), the random potential is unimportant, and an ordinary diffusion occurs after a long time: $\langle z \rangle \simeq (T/m)^{1/2} t^{1/2}$.

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