

# Gas of anyons on a lattice in the low-density limit

A. A. Belov, S. Ya. Zhitomirskaya, Yu. E. Lozovik, and V. A. Mandel'shtam  
*M. V. Lomonosov Moscow State University*

(Submitted 7 June 1990)

Pis'ma Zh. Eksp. Teor. Fiz. **52**, No. 2, 767–768 (25 July 1990)

A Green's-function method is used to derive an exact expression for the second virial coefficient of a lattice gas of particles with fractional statistics. Such a gas arises in certain models of high- $T_c$  superconductors.

Considerable progress has recently been achieved in research on the properties of systems of identical particles with fractional statistics. Such systems are playing an important role in the effort to explain the mechanism for high- $T_c$  superconductivity (see the review article by Chen *et al.*<sup>1</sup>). Wilczek *et al.*<sup>2</sup> have undertaken a study of the thermodynamics of anyons. Laughlin *et al.*<sup>3</sup> have started a study of anyons in the random phase approximation. Fröhlich and Marchetti<sup>4</sup> have formulated a lattice theory of anyons. In the mean-field approximation, a lattice anyon gas reduces to the Hofstadter problem.<sup>5</sup>

The problem of a low-density anyon gas, in particular, the problem of the interaction of two anyons on a  $\mathbf{Z}^2$  lattice, is extremely complex, in contrast with the  $\mathbb{R}^2$  case.

In the present letter we find a partial solution of this problem. Specifically, we find an expression for the lattice Green's function of a particle in the field of a single

vortex. This expression can be used to calculate various characteristics of the particle-plus-vortex system and thus the thermodynamic properties of a rarefied gas of anyons. To illustrate the method, we derive an expression for the second virial coefficient  $B(\varphi, \beta)$ , where  $\varphi$  is the field of the vortex (in units of the flux quantum), and  $\beta = 1/T$ . The method of this paper can be used to construct an effective Lagrangian of a gauge theory with vortices.

The continuous analog of this problem (which was taken up in the limit  $\beta \rightarrow \infty$  by Carslaw back in 1909) has arisen at various times and independently: in the Aharonov-Bohm effect,<sup>6</sup> in polymer physics,<sup>7</sup> in the theory of anyons,<sup>2</sup> and in other applications.

We turn now to the formulation of the model. The Hamiltonian of a composite particle-plus-(Dirac string) formation is

$$H = -t \sum_{\langle ij \rangle} C_i^\dagger e^{-i\theta} C_j. \quad (1)$$

A phase  $\varphi = \pi - \theta$  (for particles with an  $\theta$  statistics) is assigned to each edge which intersects the string, and we have  $\theta_{ij} = \varphi N_{ij}$ , where  $N_{ij}$  is the number of strings which intersect edge  $\langle ij \rangle$ . In the limit of a low-density gas, it is sufficient to know the second virial coefficient

$$\tilde{B}(\varphi, \beta) = - \frac{1}{Z_1(\beta)} \tilde{Z}_2(\varphi, \beta), \quad (2)$$

where  $Z_1(\beta) = I_0^2(2\beta)$  is a single-particle partition function, and  $Z_2$  is a partition function with respect to the motion of two particles. Here and below, the tilde means a subtraction of the value of the function at  $\varphi = 0$  (to avoid divergences).

The partition function  $Z_2(\varphi, \beta)$  is related by a Laplace transform to the trace of the Green's function  $\tilde{G}_\varphi(z)$ :

$$\text{tr} \tilde{G}_\varphi(z) = \int_0^\infty e^{-\beta z} \tilde{Z}_2(\varphi, \beta) d\beta. \quad (3)$$

We will calculate the Green's function  $G$  by making use of a lattice analog of the method of a tiling space. We first consider the four-string gauge, so that the strings will bound the quadrants of the lattice. The motion in each quadrant is described by a free Hamiltonian

$$H_0 = -t \sum_{\langle ij \rangle} C_i^\dagger C_j.$$

We write Green's function (3) as a sum over an ensemble of random walks:

$$\text{tr} \tilde{G}_\varphi(z) = \sum_{\mathbf{x} \in \mathbb{Z}^2} \sum_{\gamma: \mathbf{x} \rightarrow \mathbf{x}} z^{|\gamma|} (e^{ik(\gamma)\varphi} - 1), \quad (4)$$

where  $|\gamma|$  is the length of path  $\gamma$ , and  $k(\gamma)$  is the number of times  $\gamma$  loops the vortex.

In a summation over paths we will use a version of the block-resolvent expansion.<sup>8</sup> Specifically, with each closed path  $\gamma$  we associate an enlarged path  $\Gamma$ , which

consists of the coordinates of sequential first jumps from one straight line  $\{(x=0), (y=0)\}$  to another. The problem of summing over the paths  $\gamma$ , which correspond to the path  $\Gamma(\gamma)$ , of length  $K$ , is factorized into a problem of a random walk on  $\mathbf{Z}_4$ , which leads to the given number of loops of the vortex,  $k(\gamma)$ , and a problem of finding the Green's function for a jump from the ray  $(x>0, y=0)$  to the ray  $(x=0, y>0)$  in the course of the free motion of the particle in the first quadrant, convolved with itself  $K$  times.

We proceed to the final expression for the function of interest here, putting aside a detailed mathematical derivation for another case. We introduce the integral operator  $\hat{F}_z: L_2(0,1) \rightarrow L_2(0,1)$  with the kernel

$$F_z(x, y) = \frac{2z}{\pi} \frac{E(4z\sqrt{xy})}{1 - 16z^2xy}, \quad (5)$$

where  $E(k)$  is the complete elliptic integral of the second kind. We introduce  $\hat{A}_z \equiv (2\hat{F}_z)^{-1} [1 - (1 - 4\hat{F}_z^2)^{1/2}]$ . We then have the following expression for  $G_\varphi(z)$ :

$$\text{tr } \tilde{G}_\varphi(z) = 8z(1 - \cos \varphi) \text{Tr} [\hat{A}_z^3 \frac{\partial \hat{A}_z}{\partial z} (\hat{A}_z^4 + 1) (1 - \hat{A}_z^4)^{-1} (1 - 2\hat{A}_z^4 \cos \varphi + \hat{A}_z^8)^{-1}]. \quad (6)$$

Taking inverse Laplace transforms of (6), we find  $\tilde{B}(\varphi, \beta)$  from (2) and (3).

Since the operator  $\hat{A}_z$  is compact, the function  $\tilde{B}(\varphi, \beta)$  can be found numerically with high accuracy. The results of some numerical calculations will be published separately.

We wish to thank Ya. G. Sinai for interest in this study and S. K. Nechaev for useful discussions.

<sup>1</sup> Y.-H. Chen, F. Wilczek, E. Witten, and B. I. Halperin, *Int. J. Mod. Phys. B* **3**, 1001 (1989).

<sup>2</sup> D. Arovas, J. R. Schrieffer, F. Wilczek, and A. Zee, *Nucl. Phys. B* **251**, 117 (1985).

<sup>3</sup> A. L. Fetter, C. B. Hanna, and R. B. Laughlin, *Phys. Rev. B* **39**, 9679 (1989).

<sup>4</sup> J. Fröhlich and Marchetti, *Commun. Math. Phys.* **121**, 177 (1989).

<sup>5</sup> D. R. Hofstadter, *Phys. Rev. B* **14**, 2239 (1976).

<sup>6</sup> C. C. Gerry and V. A. Singh, *Phys. Lett.* **92A**, 11 (1982).

<sup>7</sup> A. Yu. Grosberg and A. R. Khokhlov, *Statistical Physics of Macromolecules*, Nauka, Moscow, 1989.

<sup>8</sup> J. Fröhlich and T. Spencer, *Commun. Math. Phys.* **88**, 151 (1983).

Translated by D. Parsons