

Stochastic resonance in the linear and nonlinear responses of a bistable system to a periodic field

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The validity of the theory of the linear response (the satisfaction of the standard spectral relations) during a stochastic resonance is demonstrated. A stochastic resonance in relatively strong fields is studied. The reason for the appearance of a stochastic resonance and the region of parameter values in which it is seen are pointed out.

A fundamental new phenomenon in bistable and multistable systems is a stochastic resonance: an increase and a subsequent decay of the signal-to-noise ratio R with increasing level of the external noise. The signal-to-noise ratio here is the ratio of the peak height on the spectral density of the fluctuations, $Q(\omega)$, at the frequency of the external field, Ω , to the value of the spectral density of the fluctuations in the absence of a field, $Q^{(0)}(\Omega)$ (there are also other, essentially equivalent, definitions of a stochastic resonance; see Refs. 1 and 2 and the bibliography there). Stochastic resonances have been observed in ring lasers¹ and in numerical and analog simulations of various bistable systems.¹⁻³ In studies of the stochastic resonance, its nonlinear nature is usually stressed.

In the present letter we show that in a weak periodic field a stochastic resonance can be described completely by the theory of the linear response. It thus becomes simple to see the reason for the occurrence of a stochastic resonance and to determine how R depends on the parameters. For definiteness, an analysis is carried out for a Brownian particle in a two-well potential $U(q)$ (interest in this model has revived because of, in particular, research on Josephson-junction systems⁴):

$$\ddot{q} + 2\Gamma\dot{q} + U'(q) = A \cos \Omega t + f(t), \quad \langle f(t)f(t') \rangle = 4\Gamma T \delta(t - t'). \quad (1)$$

As a result of forced oscillations, which are determined at a small field amplitude A by the linear susceptibility $\chi(\Omega)$, i.e., $\langle q(t) \rangle = A \operatorname{Re}[\chi(\Omega) \exp(-i\Omega t)] + \text{const}$, a δ -function peak arises at the field frequency Ω on the spectra fluctuation density of the system,

$$Q(\omega) = \lim_{T \rightarrow \infty} (4\pi T)^{-1} \left| \int_{-T}^T dt e^{i\omega t} \dot{q}(t) \right|^2. \quad (2)$$

The ratio of its integral intensity (the area under it) to $Q^{(0)}(\Omega)$, i.e., to the value of the spectral fluctuation density at $A = 0$, is evidently

$$R = \frac{1}{4} A^2 |\chi(\Omega)|^2 / Q^{(0)}(\Omega) \quad (3)$$

$[Q(\omega)$ in (2) is equal to the Fourier transform of the correlation function $\langle q(t+t_1)q(t_1) \rangle$, averaged over t_1 ; it corresponds to the spectral density of fluctuations which is ordinarily measured experimentally]. For a system which is at thermodynamic equilibrium at $A = 0$ [or at a quasiequilibrium if the noise $f(t)$ is of a non-thermal nature], $\chi(\omega)$ can be expressed in terms of $Q^{(0)}(\omega)$ in the standard way:

$$\text{Re } \chi(\omega) = \frac{2}{T} \int_0^\infty d\omega_1 Q^{(0)}(\omega_1) \omega_1^2 (\omega_1^2 - \omega^2)^{-1}, \quad \text{Im } \chi(\omega) = \frac{\pi\omega}{T} Q^{(0)}(\omega). \quad (4)$$

The quantitative agreement between the values of R measured⁵ through an analog simulation of system (1), on the one hand, and the values of R calculated from Eqs. (3) and (4) with the help of the spectral density of fluctuations $Q^{(0)}(\omega)$ measured for the same system at $A = 0$, on the other, is demonstrated by Fig. 1. These results show that the fluctuation-dissipation theorem and the dispersion relations hold under conditions corresponding to a stochastic resonance.

The appearance of a stochastic resonance with a periodic modulation⁶ of the probabilities for fluctuational interwell transitions W_{ij} and of the populations of the potential wells, w_i ($i = 1, 2$), by an alternating field (cf. Ref. 1) [*sic*]. In the absence of a field, fluctuational transitions at $T \ll \Delta U_{1,2}$ (ΔU_i is the depth of well i) lead to a narrow peak $Q_{tr}^{(0)}(\omega)$ in $Q^{(0)}(\omega)$ (Ref. 7):

$$Q_{tr}^{(0)}(\omega) = \frac{1}{\pi} w_1 w_2 (q_1 - q_2)^2 W / (W^2 + \omega^2), \quad W = W_{12} + W_{21} \quad (5)$$

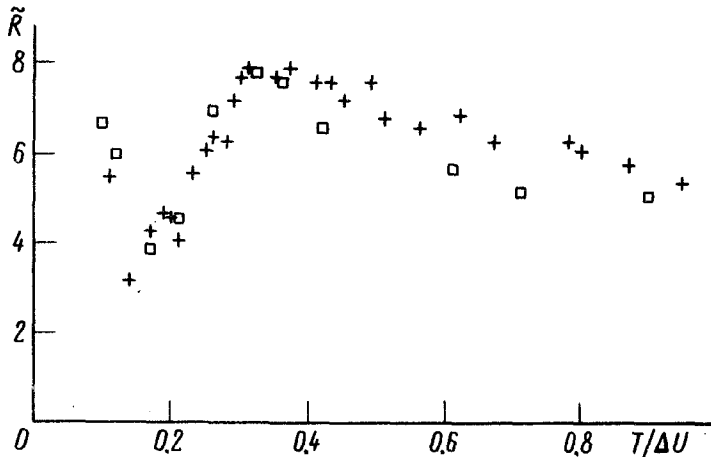


FIG. 1. Values of $\tilde{R} = 6.51 \times 10^{-4} R$ for the potential $U(q) = -\frac{1}{4}q^2 + \frac{1}{4}q^4$; $\Omega = 0.0695$, $A = 0.1$, $\Gamma = 0.125$; $\Delta U = 1/4$. \square —Direct measurements; $+$ —calculated from measured values of $Q^{(0)}(\omega)$.

($q_{1,2}$ are equilibrium positions). If only the term in (5) is retained in $Q^{(0)}(\omega)$, it is clear from (3) and (4) that we have

$$R = \frac{1}{4} \pi A^2 (q_1 - q_2)^2 w_1 w_2 W / T^2 \propto \exp(-\Delta U_{max}/T), \quad \Delta U_{max} = \max(\Delta U_1, \Delta U_2). \quad (6)$$

In other words, the signal-to-noise ratio R increases exponentially with increasing T , both for equal well depths¹⁻³ and for different well depths.

The range of applicability of (6) and of the existence of a stochastic resonance is found from the condition that $Q^{(0)}(\omega)$ is close to $Q^{(0)}_{tr}$, i.e., the condition that the parts of $Q^{(0)}(\omega)$ due to vibrations with respect to the equilibrium positions, $Q_i^{(0)}(\omega) \approx 2/\pi \Gamma T w_i [(\omega^2 - \omega_i^2)^2 + 4\Gamma^2 \omega^2]^{-1}$, where $\omega_i = [U''(q_i)]^{1/2}$ ($T \ll \Delta U_i$) are small. Since $W_{ij} \propto \exp(-\Delta U_i/T) \ll 1$, a stochastic resonance is clearly manifested only at very low frequencies $\Omega \ll \omega_{1,2}, \omega_{1,2}^2/\Gamma$. In systems with a slight damping, $\Gamma \ll \omega_{1,2}$ the increment $\delta Q_i^{(0)} \approx (4\Gamma^2 + \omega^2)^{-1} \Gamma/2\pi w_i [TU'''(q_i)\omega_i^{-4}]^2$ becomes important⁷ in the region $\omega \lesssim 2\Gamma$, even at comparatively small values of T . This circumstance seriously limits the interval of Ω and T values in which a stochastic resonance is observed, as was verified in the present experiments. It is clear from this discussion that for a given Ω and for very small values of T , under the condition $W \ll \Omega$, the ratio R decreases with T (as $1/T$ at $\Gamma \gtrsim \omega_{1,2}$). At large T , an increase in R then occurs, as was observed in Ref. 1 (cf. Fig. 1).

At low values of T , the response of a bistable system to a low-frequency field ($\Omega \ll \Gamma, \omega_{1,2}^2/\Gamma$) is very nonlinear even at small values of A , under the conditions $\Delta U_{1,2} \gg A|q_{1,2} - q_s| \gg T$ [q_s is the position of a local maximum of $U(q)$], since the modulation of the well populations due to the field is strong. Solving the balance equations for the populations, and incorporating the periodic increment in ΔU_i in the expression for the transition probability $W_{ij} \propto \exp(-\Delta U_i/T)$, we can show that the term associated with the transitions, $\langle q(t) \rangle_{tr}$, in $\langle q(t) \rangle$ described by a nearly rectangular crest. In the simplest case of a symmetric potential, $U(q) = U(-q), q_2 = -q_1$, we would have

$$\langle q(t) \rangle_{tr} \approx 2\bar{q} \sum_{n=-\infty}^{\infty} \left[\theta\left(t - \frac{2\pi n}{\Omega}\right) - \theta\left(t - \frac{\pi(2n+1)}{\Omega}\right) \right] - \bar{q},$$

$$\bar{q} = q_1 \tanh \lambda, \quad \lambda = \left(\frac{2\pi T}{|Aq_1|} \right)^{1/2} \frac{W_{12}}{2\Omega} \exp\left(\frac{|Aq_1|}{T} \right) \quad (W_{12} = W_{21}). \quad (7)$$

Under the condition $\lambda \gg 1$ (note that we have $1 - \tanh \lambda < 0.1$ for $\lambda \gtrsim 1.5$), we have $\bar{q} \approx q_1$, and \bar{q} is essentially independent of T . The peak intensity $Q(\omega)$ in (2) at the frequency Ω is thus also nearly independent of T . A stochastic resonance arises in the region $\Omega < W = 2W_{12}$ (in which the condition $\lambda \gg 1$ definitely holds). It results from a decrease in $Q^{(0)}(\Omega)$ with increasing T , as a result of a spreading of the peak in (5): $Q_{tr}^{(0)}(\Omega) \propto \exp(\Delta U_1/T)$ at $\Omega \ll W$. It can be seen from (2) and (7) that the $Q(\omega)$

peaks at the overtones of Ω were found to be small in the regime of a strong nonlinearity in Refs. 2 and 3 because of a numerical reason: The intensity of the peak at the frequency $(2k + 1)\Omega$ is proportional to $(2k + 1)^{-2}$.

It follows from the results of the present study and those of Refs. 6 and 7 that a stochastic resonance should occur both at low frequencies and at frequencies close to the frequency of the strong field in systems with several stable forced-oscillation regimes in an intense periodic field (optically bistable and multistable systems, electrons excited by a field at the cyclotron frequency,⁸ etc.).

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