Multivalued action functionals, Lorentz harmonics, and spin

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The quantization of the helicity of massless particles (end of supersymmetric multiplets) is shown to be a consequence of a multivaluedness of the action functional of a new twistor-harmonic formulation. A new and purely topological proof of the quantization of the helicity in units of $\hbar/2$ is given.

Models of point and extended entities in spaces and superspaces containing additional harmonic[1] and twistor variables have recently been the subject of active research in connection with the problem of a covariant quantization of superparticles and superstrings.[2-9] It was shown in Ref. 9, for example, that the familiar problem of the covariant separation of Grassmann constraints for superparticles into irreducible constraints of the first and second kinds has a very simple solution in $D=4$ if the boson spinor variables $v_\alpha^{\mp} \equiv v_\alpha^{(0|\mp 1)}$, $\bar{v}_{\dot\alpha}^{\pm} \equiv \bar{v}_{\dot\alpha}^{(\pm 1|0)} = (\overline{v_\alpha^{\mp}})$, are included among the coordinates of the superspace. These variables are constrained by the conditions $\Xi \equiv v^{\alpha -} v_\alpha^{+} - 1 = 0$, $\bar{\Xi} \equiv \bar{v}_{\dot\alpha}^{-} \bar{v}^{\dot\alpha +} - 1 = 0$ (Ref. 1) and are defined to within transformations from the local (in the space $\{v_\alpha^{\mp}, \bar{v}_{\dot\alpha}^{\pm}\}$) group $[U(1)]^c \approx U_L(1) \otimes U_R(1) \approx SO(1,1) \otimes SO(2)$ $[q_R(v_\alpha^{\mp}) = -q_L \times (\bar{v}_{\dot\alpha}^{\pm}) = \mp 1;\ q_L(v) = q_R(\bar{v}) = 0)]$. A formulation of the theory of zero-super-p-branes,[10] which generalizes Ref. 9, was constructed in Ref. 11, where it was shown that there are no obstacles to a BRST–BFV quantization in accordance with a BFF scheme.[12]

Our purposes in the present letter are to call attention to a relationship between the spin and the topology of the twistor-harmonic sector which is manifested in such models and to carry out a new and purely topological proof of the quantization of the helicity of massless particles (and of supermultiplets) in units of $\hbar/2$.

For a massless particle moving in the space $M^{1+3} \otimes (SL(2,\mathbb{C})/[U(1)]^c$ ($M^{1+3} \equiv \{x^m\}$ is Minkowski space, $SL(2,\mathbb{C})/[U(1)]^c \equiv \{v_\alpha^{\mp}, \bar{v}_{\dot\alpha}^{\pm}\}$; Ref. 9), the action is[9]

$$S = S_0 + S_{WZ} \ , \tag{1a}$$

$$S_0 = -\frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau n^{(-|-)} (v^- \sigma_m \bar{v}^+) \partial_\tau x^m , \tag{1b}$$

$$S_{WZ} = -l_1 \int_{\tau_i}^{\tau_f} d\tau \Theta_\tau - l_2 \int_{\tau_i}^{\tau_f} d\tau \bar{\Theta}_\tau , \tag{1c}$$

where τ is the proper time, and $\Theta = \overline{(\bar{\Theta})} = 1/2i (v^{\alpha -} dv_\alpha^{+} + v^{\alpha +} dv_\alpha^{-}) = d\tau \Theta_\tau$ is the

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Cartan theta form[13] for $SL(2,\mathbb{C})/[U(1)]^c$ (Ref. 9). The quantity S_0 in (1b) is a twistor-like action functional. The quantity S_{WZ} is a sum of Weiss–Zumino (topological) terms. A supersymmetric generalization of action (1) can be carried out by making the following substitution in (1b): $\partial_\tau x^m \rightarrow \omega_\tau^m = \partial_\tau x^m - i\partial_\tau \theta_i^\alpha \sigma_{\alpha\dot\alpha}^m \bar\theta^{\dot\alpha i} + i\theta_i \sigma^m \partial_\tau \bar\theta^i$.

The quantum state vector of model (1) describes a particle with a helicity[1)] (Ref. 9) $s = 1/2(\tilde l_1 + \tilde l_2)$. Here l_1 and l_2 are quantum ("renormalized") values of the coefficients l_1 and l_2 of the Wess–Zumino terms in (1c), which may differ from the classical values of l_1 and l_2 by a sum of ordering constants:

$$\tilde l_1 = l_1 + \textit{sum of ordering constants}, \quad \tilde l_2 = l_2 + \textit{sum of ordering constants}. \tag{2}$$

There is thus the possibility that Wess–Zumino terms will be induced by effects of an ordering of quantum operators (or vacuum loops, in Feynman-diagram terms) which are analogous to the incorporation of a Casimir vacuum energy. In this sense one can say that the Weiss–Zumino terms are associated with anomalies.[2)]

A quantization of the helicity $s = 1/2(\tilde l_1 + \tilde l_2)$ in units of $\hbar/2$ was found in Ref. 9 through the solution of a differential equation for the state vector. An important point to note, however, is that this fact can be proved on the basis of purely topological considerations, just as the quantization of the charge of a magnetic monopole was proved in Ref. 14 (see also Ref. 15).

The sum of Wess–Zumino terms in (1c) is invariant under $[U(1)]^c \approx U_L(1) \otimes U_R(1)$ gauge transformations only to within "boundary" terms $U_L(1) \approx \{\exp[i(\gamma(\tau) + i\beta(\tau))]\}$, $U_R(1) \approx \{\exp[i(\gamma - i\beta)]\}$

$$S'_{WZ} = S_{WZ} - (l_1 + l_2)\int_{\tau_i}^{\tau_f} d\gamma(\tau) - i(l_1 - l_2)\int_{\tau_i}^{\tau_f} d\beta(\tau). \tag{3}$$

The identification of $[U(1)]^c$ transformations at the initial and final moments of the proper time $\{\exp[i(\gamma(\tau_i) + i\beta(\tau_i))] = \exp[i(\gamma(\tau_f) + i\beta(\tau_f))]\}$ leads to the following conditions on the parameters of the $SO(1,1)$ groups: $\beta(\tau_i) = \beta(\tau_f)$ and $SO(2) \approx U(1): \gamma(\tau_i) - \gamma(\tau_f) = 2\pi k$. The second term in (4) disappears, and the first reduces to $2\pi k(l_1 + l_2)$:

$$S'_{WZ} = S_{WZ} + 2\pi k(l_1 + l_2), \quad (k \in \mathbb{Z}). \tag{4}$$

The integer k in (4) is a topological invariant: the degree of the mapping of the space of the compactified proper time S_τ^1 into the group space S^1 of subgroup $U(1) \subset [U(1)]^c$ $(k = 1/2\pi \int_{S_\tau^1} d\gamma \in \pi_1(S^1) \approx \mathbb{Z}$; $\gamma(\tau)$: $S_\tau^1 \rightarrow S^1 \approx SO(2) \approx U(1) \subset [U(1)]^c$. Consequently, functional (1) is invariant only under topologically trivial transformations from $U(1) \subset [U(1)]^c$. However, $[U(1)]^c$ transformations are equivalence transformations on the set of spinor boson variables $v_\alpha^\mp, \bar v_{\dot\alpha}^\pm$, which determine the harmonic sector of our space:[9] $SL(2,\mathbb{C})/[U(1)]^c$. Consequently, the invariance in (4) of action (1) under topologically nontrivial transformations from $U(1) \subset [U(1)]^c$ should be interpreted as a multivaluedness of the action functional:[14,15]

$$S \sim S + 2\pi k(l_1 + l_2), \quad (k \in \mathbb{Z}). \tag{5}$$

In other words, action (1) is defined only modulo $2\pi k(l_1 + l_2)$.

Additive increments as in (2) may be found[9] during the quantization of the constants l_1 and l_2 as a result of ordering constants. Consequently, by virtue of (1) and (5), the effective action of the quantum theory is defined only modulo $2\pi k(\tilde{l}_1 + \tilde{l}_2)(k \in \mathbb{Z})$:

$$S_{eff} \sim S_{eff} + 2\pi k(\tilde{l}_1 + \tilde{l}_2). \tag{6}$$

However, the measure of the path integration containing the factor $\exp\{(i/\hbar)S_{\rm eff}\}$ must be defined unambiguously: $\exp\{(i/\hbar)S_{\rm eff}\} = \exp\{i/\hbar[S_{\rm eff} + 2\pi k(\tilde{l}_1 + \tilde{l}_2)]\}$. This requirement leads to a quantization of the helicity in units of $\hbar/2$ also:

$$s = \frac{1}{2}(\tilde{l}_1 + \tilde{l}_2) = 0; \pm \frac{\hbar}{2}; \pm \hbar; \pm \frac{3}{2}\hbar; \ldots . \tag{7}$$

The quantization of the helicity of massless particles (and supermultiplets) in (7) is therefore a consequence of the multivaluedness in (6) of the effective action functional and has been proved on the basis of purely topological considerations.

1) The constants l_1 and l_2 in (1c) have the dimensionality of an action, since the harmonics are dimensionless by virtue of the "harmonicity" conditions $\Xi = v^{\alpha -} v_\alpha^+ - 1 = 0$.

2) In a twistor formulation of the theory of (super-) particles[8] similar to that of Ref. 9, Wess–Zumino terms are not incorporated in the action. In other words, from our point of view, these terms can only be induced upon quantization as a result of ordering effects: $(\tilde{l}_1 + \tilde{l}_2)$ (Ref. 8) $\equiv c$ (Ref. 8) = sum of ordering constants. It is for this reason that the value of the spin in Ref. 8 was represented as being bounded from above by the number of variables in the theory (twistors).

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