Complex-matrix model and 2D quantum gravity

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(Submitted 23 July 1990)
Pis'ma Zh. Eksp. Teor. Fiz. **52**, No. 5, 885–887 (10 September 1990)

An exact solution of the model of complex $N \times N$ matrices is proposed. The "string equation" corresponding to 2D quantum gravity is reproduced in the continuum limit.

Recent progress[1–4] in the theory of noncritical strings and 2D quantum gravity has been based on a model of Hermitian matrices, which has served as a convenient discretization of the continuum theory and which can be solved in the scaling limit. Alternative matrix models have also been examined. Periwal and Shevits[5] have shown that the unitary matrix model reduces to a Painlevé II equation and belongs to a different universality class.

In the present letter we examine a model of $N \times N$ complex matrices, determined by the partition function

$$Z = \int \prod_{i,j=1}^{N} d\mathrm{Re}\varphi_{i,j} d\mathrm{Im}\varphi_{i,j} \exp\{-N \sum_{j \geq 1} \frac{1}{j} g_j \mathrm{tr}(\varphi^+ \varphi)^j\}. \tag{1}$$

This model corresponds to a discretization of random surfaces by oriented polygons. To solve it, we use a contour equation method which has been used previously[6–9] for a model of Hermitian matrices. Contour equations for the complex-matrix model with $N = \infty$ were solved in Ref. 10. At finite N, it is convenient to write the contour equation for the Laplace transform of a Wilson loop:

$$G(p) = < \frac{1}{N} \mathrm{tr} \frac{p}{p^2 - \varphi^+ \varphi} >, \tag{2}$$

where the expectation value is defined in accordance with (1). The contour equation follows from the invariance of the measure and can be written

$$\int_C \frac{d\omega}{2\pi i} \frac{g(\omega)}{p - \omega} G(\omega) = (G(p))^2 + \frac{1}{2N^2} \frac{\partial}{\partial p} \frac{\delta}{\delta g(p)} G(p), \tag{3}$$

where the integration contour C includes the singularities of $G(\omega)$,

$$g(p) = \sum_{j=1}^{\infty} g_j p^{2j-1} \quad \text{and} \quad \frac{\delta}{\delta g(p)} = \sum_{j=1}^{\infty} \frac{1}{p^{2j}} \frac{\partial}{\partial g_j}. \tag{4}$$

The meaning of this equation is that $-g_j/j$ serves as sources[7] for the fields $\mathrm{tr}(\varphi^+\varphi)^j$. The functional $-2\int_0^p dp g(p)$ constructed from them is a source for a Wilson loop. The associated correlation functions M of the loops are found from $G(p)$ through an $(M-1)$-fold application of $1/2\partial/\partial p_i \delta/\delta g(p_i)$. The "extraneous" g_j should then be set equal to zero. A chain of equations for M-loop expectation values is found from Eq. (3) by a variational method. Equation (3) should be supplemented with the boundary condition

$$pG(p) \to \lambda \quad \text{as} \quad p \to \infty, \tag{5}$$

where it is convenient to assume $\lambda \neq 1$. Taking the limit $p \to \infty$ in Eq. (3), we can put condition (5) in the form

$$2\lambda = \int\limits_C \frac{d\omega}{2\pi i} g(\omega)\omega \dot{G}(\omega), \tag{6}$$

where $\dot{G} \equiv \partial G/\partial \lambda$.

Equation (3) can be solved by iterations in $1/N^2$. In the leading order, the result is the same as in the case of the model of Hermitian matrices with an even potential:

$$G_0(p) = \frac{1}{2}\int\limits_C \frac{d\omega}{2\pi i}\frac{g(\omega)}{p-\omega}\frac{\sqrt{p^2-z}}{\sqrt{\omega^2-z}}, \tag{7}$$

where z is found from boundary condition (6). To prove (7) and to find z, we note that by explicitly differentiating (7) we find

$$\dot{G}_0(p) = 1/\sqrt{p^2-z}. \tag{8}$$

Expressions for the coupled M-loop correlation function in the leading order in $1/N^2$ can be found through a variation of expression (7). For $M=2$ we find

$$\chi_0^{(2)}(p,q) = \frac{\partial}{2\partial q}\frac{\delta}{\delta g(q)} G_0(p) = \frac{1}{4(p^2-q^2)^2}\left[\frac{2p^2q^2 - zp^2 - zq^2}{\sqrt{p^2-z}\sqrt{q^2-z}} - 2pq\right], \tag{9}$$

in agreement with Ref. 7.

The structure of the solution in the leading order and that of the first correction in terms of $1/N^2$ suggest that the exact result for $\dot{G}(p)$ is given by the diagonal resolvent of a Sturm–Liouville operator:

$$\dot{G}(p) = 2R[p,u] \equiv 2 < \lambda \mid (p^2 - \frac{1}{N^2}\frac{\partial^2}{\partial\lambda^2} + u(\lambda))^{-1} \mid \lambda >, \tag{10}$$

where $u(\lambda)$ is found through substitution into Eq. (6). That equation then takes the form

$$\lambda = \sum_{j\geq 1} g_j R_j[u], \tag{11}$$

Here $R_j[u]$ represents generalized Korteweg–de Vries potentials, which arise in the Gel'fand–Dikiĭ expansion[11]

$$R[p,u]=\sum_{j=0}^{\infty}R_j[u]/p^{2j+1}. \tag{12}$$

Since we have $R_1[u] = -u/4$, the quantity $-u/2$ represents the heat capacity of the matrix model. In the leading order in $1/N^2$ we have $u = -z$.

Equation (11) is the same as the generalized string equation of Ref. 3, which was derived in the continuum limit of the Hermitian-matrix model. To take the continuum limit in Eq. (11), we must set[1] $\lambda \to \lambda_c - a^m\Lambda$ near the multicritical point of order m, for which we have $z_c - z \sim (\lambda_c - \lambda)^{1/m}$. The equations found in the double scaling limit ($a \to 0$, $N \to \infty$ at a fixed N^2a^{2m+1}) are the same as the corresponding equations of Refs. 1–4, so the models of Hermitian and complex matrices belong to a common universality class. However, the complex-matrix model has the advantage that it has an exact solution at a finite N, not exclusively in the scaling limit.

We have not been able to rigorously prove that ansatz (10), (11) solves Eq. (3) at finite N. There are many arguments in addition to those above in favor of this proposition. At any rate, our assumption is equivalent to that of Ref. 8 regarding the structure of the solution of the continuum equations for the Hermitian-matrix model.

On the basis of the first correction in terms of $1/N^2$ which has been calculated, it can be asserted that in the $m = 2$ continuum limit, which corresponds to a massless 2D quantum gravity, the model of complex matrices leads to a Painlevé I equation. We hope that the model of complex matrices, which has an exact solution at finite N, may prove useful in a study of the problem raised in Ref. 6, concerning real nonperturbative solutions of 2D quantum gravity.

I wish to thank J. Ambjørn, A. Gerasimov, A. Losev, A. Morozov, and A. Perelomov for useful discussions.

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Translated by D. Parsons