New conformal theories, $k=4$ magic level, and integrable models

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A $(D=d-2)$-parameter family of conformal theories is derived through a generalized GKO construction of Morozov *et al.* for an $SO(d)_4$ algebra. This family is specified by D algebraic equations of degree $3D$ in a $2D$ space of integrable models.

One of the fundamental unresolved problems which have arisen in connection with string theory is that of classifying the conformal theories, in particular, searching for multiparameter families of QFTs.

Morozov *et al.*[1] have recently emphasized a profound and not completely clear relationship between RQFTs and Turbiner's quasi-exactly solvable models in quantum mechanics. In addition, Morosov *et al.*[1] derived equations for the coefficients C^{ab} of a generalized Sugawara construction

$$T(z)=C^{ab}J_a(z)J_b(z)\ , \tag{1}$$

which follow from the Virasoro conditions on $T(z)$.

Our purpose in the present letter is to demonstrate that multiparameter solutions of the Virasoro conditions actually exist. In the case of an $SO(d)_k$ algebra with $k=4$, the conformal theories form a D-dimensional surface $(D=d-2)$ in the R^{2D} space of parameters of integrable models. We are thus looking at the relationship between QFTs and exactly solvable models from a slightly different aspect.

The substitution of a diagonal ansatz[1] for $SO(d)_k$,

$$C_{\alpha\beta,\gamma\delta}=C_0S_{\alpha\beta}\ \delta^{\alpha\beta,\gamma\delta}\ , \tag{2}$$

into the system of Virasoro conditions leads to a homogeneous system of cubic equations for $S_{\alpha\beta}$:

$$\sum_{\alpha\neq 1,2,3}\{S_{12}S_{13}(S_{2\alpha}-S_{3\alpha})+S_{1\alpha}(S_{12}S_{3\alpha}-S_{13}S_{2\alpha})\}$$
$$=(k/2-1)S_{12}S_{13}(S_{13}-S_{12})+S_{23}(S_{13}^2-S_{12}^2), \tag{3}$$

where the indices 1, 2, 3 can be replaced by any trio δ, β, $\gamma\in\{1,\dots,d\}$. Since the number of independent equations of system (3), which is $[d(d-1)/2]-1$, is the same as the number of coefficients $S_{\alpha\beta}$, which are determined within a scale factor, system (3) generally has only a discrete set of solutions, namely, Sugawara solutions and cosets. However, there exists an unusual value, $k=4$, for which multiparameter families of solutions are possible.

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With $k = 4$ it follows from (3) that for each quartet of indices $\alpha, \beta, \gamma \in \{1, \dots, d\}$ there exists a constraint

$$(S_{\alpha\beta} - S_{\alpha\gamma})(S_{\beta\delta} S_{\gamma\delta} S_{\alpha\delta} S_{\beta\gamma}) + (S_{\alpha\gamma} - S_{\beta\gamma})(S_{\alpha\delta} S_{\beta\delta} + S_{\gamma\delta} S_{\alpha\beta}) + (S_{\beta\gamma} - S_{\alpha\beta})(S_{\alpha\delta} S_{\gamma\delta} + S_{\beta\delta} S_{\alpha\gamma}) = 0. \quad (4)$$

It is a straightforward matter to show, by direct substitution, that the Manakov ansatz[3] is a solution of system (4):

$$S_{\alpha\beta} = (a_\alpha - a_\beta)/(b_\alpha - b_\beta). \quad (5)$$

After the reparametrization degrees of freedom are eliminated (say, after we have set $a_1 = b_1 = 0$, $a_2 = b_2 = 1$), we are thus left with $2D$ independent parameters. As was shown in Ref. 3, dynamic systems on SO(d) with the Hamiltonian

$$H = \sum_{\alpha\beta} S_{\alpha\beta} J^{\alpha\beta} J^{\alpha\beta}, \quad (6)$$

are completely integrable. The QFTs described by the generalized GKO construction[1] can thus be characterized by means of an algebraic manifold M_{VIR} specified mod(4) by system (3) in the space of parameters of integrable models (6).

We can now show that $\dim M_{VIR} = D$. We consider a set of three equations (3), differing by permutations of the indices 1, 2, 3. In general, two of these equations are independent, but with the "magic" value $k = 4$ one of the two is a corollary of the other and of constraint (4). Consequently, system (3) identifies a particular subspace in the space R^{2D} of the parameters on which the solutions of (4) depend. The dimensionality of this subspace is equal to the maximum number of unordered pairs $(\alpha\beta)$ $(\alpha, \beta, \in \{1, \dots, d\})$ which are of such a nature that it is not possible to form from them even a single trio $(\alpha\beta)$, $(\beta\gamma)$, $(\gamma\alpha)$. Making use of the reparametrized degree of freedom, we find $\dim M_{VIR} = (d-1) - 1$. The manifold of series satisfying the Virasoro condition is thus determined by a system of D algebraic equations of degree $3D$ from $2D$ variables.

The cosets correspond to infinitely remote points in the parameter space $\{\alpha_i, \beta_i\}$ of integrable models. A question which remains open[1] is which "internal" property of the integrable models corresponds to the possibility of "elevating" them to the level of conformal theories.

The problem of classifying integrable models in semisimple Lie algebras with quadratic Hamiltonians is now basically solved.[4] It is possible that this classification holds the key to an enumeration of multiparameter families of conformal theories. The manifold of theories is apparently stratified and decays into an unconnected sum of $d-1$ components of dimensionality $m = 0, 1, \dots, D$ which are deformations of coset models $\mathrm{SO}(d)_4/\mathrm{SO}(m+1)_4$ with a central charge $C = d - m - 1$.

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