Yet another component on the order of $\alpha^2(Z\alpha)E_F$ in the hyperfine splitting in muonium and hydrogen

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(Submitted 30 July 1990)
Pis'ma Zh. Eksp. Teor. Fiz. **52**, No. 6, 937–939 (25 September 1990)

A component of the hyperfine splitting on the order of $\alpha^2(Z\alpha)E_F$ induced by a polarization insertion into a "radiation" photon is calculated. This component turns out to be $(-0.310742...)[\alpha^2(Z\alpha)/\pi]E_F$ or, numerically, -0.17 kHz for muonium and -0.054 kHz for hydrogen.

We have recently undertaken a calculation of all the purely radiative corrections on the order of $\alpha^2(Z\alpha)E_F$ to the hyperfine splitting in muonium and hydrogen[1)] (Refs. 1–3). It has been shown that all the components on this order are generated by six gauge-invariant sets of diagrams with two external photons. Corrections induced by insertions of the polarization operator in the external photon lines were calculated in Refs. 1–3. They correspond to three gauge-invariant sets of diagrams. In the present letter we calculate the last component, which is associated with insertions of the vacuum polarization. The polarization insertion in this case is made in a "radiation" photon (Fig. 1), in contrast with Refs. 1–3.

A compact expression for the sum of single-loop radiative corrections to the electron line, accompanied by the emission of two photons, was derived in Refs. 4 and 5. Only one of the matrix structures $\langle\gamma\hat{k}\gamma\rangle$ given there contributes to the hyperfine splitting in the kinematics of the external field. That particular structure is the same as the skeletal structure. The coefficient of this structure, $(\alpha/2\pi)L(k)$, was used in Refs. 1–3. The general expression from Refs. 4 and 5 was simplified slightly through an integration by parts. The insertion of a polarization operator in a radiation photon can be dealt with by reconstructing the mass of the radiated photon in the expression for the electron factor $L(k)$. The new radiation factor is then

$$\frac{\alpha}{2\pi}L(k)=\frac{\alpha^2}{2\pi^2}\int_0^1 dv\frac{v^2(1-v^2/3)}{1-v^2}L(k,\lambda), \qquad (1)$$

where $\lambda^2=4/1(1-v^2)$, and all masses and momenta are expressed in units of the electron mass.

As was shown in Refs. 1 and 2, the component of the hyperfine splitting induced by the radiative corrections to the electron line is given by the integral

$$\delta E=\frac{4\alpha(Z\alpha)}{\pi^2}E_F\int_0^\infty dk\ L(k)=\frac{4\alpha^2(Z\alpha)}{\pi^3}E_F\int_0^\infty dk\int_0^1 dv\frac{v^2(1-v^2/3)}{1-v^2}L(k,\lambda). \qquad (2)$$

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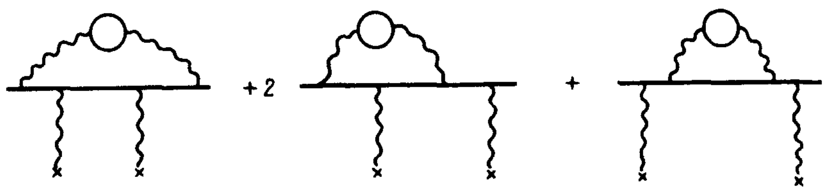

FIG. 1. Gauge-invariant set of diagrams with the insertion of a polarization operator in a "radiation."

After some lengthy calculations, we find a representation of the hyperfine splitting in the form of a one-dimensional integral:

$$\delta E = \frac{\alpha^2(Z\alpha)}{\pi} E_F \left\{ \frac{2}{\pi} \int_0^1 dq G(q) \left[\left(\arctan \sqrt{\frac{2q}{1-q}} - \frac{\sqrt{2q(1-q)}}{1+q} - \frac{4}{3} \frac{q\sqrt{2q(1-q)}}{(1+q)^2} \right) \right.\right.$$
$$\times \left(-\frac{1}{2q^3} + \frac{1}{2q} + \frac{5}{4} - \frac{1}{2(1+q)} - \frac{4}{(1+q)^2} \right)$$
$$\left.\left. + \sqrt{\frac{2q}{1-q}} \left(-\frac{4}{3q} - \frac{8}{3} + \frac{28}{3(1+q)} + \frac{8}{3(1+q)^2} - \frac{16}{(1+q)^3} + \frac{32}{3(1+q)^4} \right) \right] \right\}, \tag{3}$$

where

$$G(q) = q \int_0^{\pi/2} d\varphi \frac{\sin^2\varphi \left(1 - \frac{\sin^2\varphi}{3}\right)}{\sqrt{1 - q^2 \sin^2\varphi}}. \tag{4}$$

It is a simple matter to express the function $G(q)$ in terms of the standard complete elliptic integrals[6] $D(q)$ and $C(q)$:

$$G(q) = \frac{q}{9}(7D(q) - C(q)), \tag{5}$$

However, integral representation (4) is more convenient for calculations.

Expression (3) can be written in more compact form by eliminating the complete elliptic integral $C(q)$. For this purpose we use a representation of $C(q)$ in terms of $D(q)$ and its derivatives. Integrating by parts several times, and making use of the well-known hypergeometric equation for the complete elliptic integral[6] $D(q)$, we find the expression

$$\delta E = \frac{\alpha^2(Z\alpha)}{\pi} E_F \left\{ -\frac{149}{270} + \frac{2}{9\pi} \int_0^1 dq D(q) \right.$$

$$\left. \times \left[\frac{3}{1+q} \arctan \sqrt{\frac{2q}{1-q}} + \sqrt{\frac{2q}{1-q}} \left(-\frac{5}{4} \cdot \frac{1}{1+q} - \frac{2927}{2400} + \frac{10169}{3600} q \right) \right] \right\}. \tag{6}$$

All our efforts have failed to derive a closed expression for integral (4) [or(6)]. We have settled for a numerical integration, which can be carried out with arbitrary accuracy.

The component of the hyperfine splitting stemming from the diagrams in Fig. 1 is thus

$$\delta E_{\mathrm{Mu}} = -0,310742\ldots \cdot \frac{\alpha^2(Z\alpha)}{\pi} E_F^{\mathrm{Mu}} \cong -0,17\,\mathrm{kHz}$$

$$\delta E_{\mathrm{H}} = -0,310742\ldots \cdot \frac{\alpha^2(Z\alpha)}{\pi} E_F^{\mathrm{H}} \cong -0,054\,\mathrm{kHz} \qquad (7)$$

(for muonium and hydrogen, respectively). The sign in (7) could of course have been predicted beforehand. The reason is that the diagrams in Fig. 1 differ only by the insertion of the vacuum polarization from the diagrams for the classical correction on the order of $\alpha(Z\alpha)E_F$ (Refs. 7 and 8). It is well known that a polarization insertion does not alter the sign of the photon propagator, so the sign in (7) is the same as that of the correction of Refs. 7 and 8.

We are presently calculating the contribution on the order of $\alpha^2(Z\alpha)E_F$ from the two remaining gauge-invariant sets of diagrams.[1-3] The calculation of those corrections will make it possible to reduce the error in the theoretical expression for the hyperfine splitting in muonium to the level of the experimental error in this quantity.

[1)] E_F is the energy of the Fermi hyperfine splitting.

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Translated by D. Parsons