Spontaneous formation of autosolitons in stable nonequilibrium systems

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(Submitted 13 August 1990)
Pis'ma Zh. Eksp. Teor. Fiz. **52**, No. 6, 945–949 (25 September 1990)

The probability per unit time for the spontaneous formation of a highly nonequilibrium localized region, i.e., an autosoliton, in a stable system slightly away from equilibrium is derived. The shape of a finite-amplitude critical fluctuation, whose growth leads to the appearance of an autosoliton, is found.

A brief and local external perturbation can excite an autosoliton in many stable nonequilibrium distributed systems (semiconductor plasmas, gaseous plasmas, the ionosphere, composite superconductors, nonlinear optical systems, etc.).[1] In real systems, an autosoliton can also arise spontaneously, as the result of a local increase of an avalanche nature in the parameters of the system near a small inhomogeneity.[2,3] This avalanche increase occurs when the level (A), at which the system is being excited, is close to the point at which the homogeneous state of the system becomes unstable. In the present letter we derive the probability for the spontaneous formation of an autosoliton in nonequilibrium systems whose initial state is stable with respect to small perturbations of the parameters of the system.

The properties of many nonequilibrium systems in which autosolitons form are described by equations of the type[1]

$$\frac{\partial Y_j}{\partial t} = \sum_{i=0}^{N} \nabla (D_{ij} \nabla Y_i) - q_j(Y_0, \ldots Y_i, \ldots Y_N, A) + \xi_j(\vec{r}, t), \quad j = 0, 1, \ldots N, \tag{1}$$

where A is the level at which the system is being excited, Y_j are the parameters of the system, $D_{ij}(Y_i)$ are diffusion coefficients, q_j are nonlinear functions, and $\xi_j(\vec{r},t)$ are

 0021-3640/90/180326-05$01.00

random forces, which for definiteness are assumed to be δ-correlated: $\overline{\xi_j(\vec{r}_1,t_1)\xi_j(\vec{r}_2,t_2)} = \Phi_j\delta(\vec{r}_1 - \vec{r}_2)\delta(t_1 - t_2)$.

The stability of the homogeneous state of nonequilibrium systems which can be described by Eqs. (1) with respect to inhomogeneous perturbations of the type $\delta Y_j \propto \exp(i\vec{k}_c\vec{r})$ with a wave number $k_c \equiv |\vec{k}_c| \neq 0$ can be disrupted when the level to which the system is being excited, A, exceeds a certain critical value A_c (Ref. 3). For simplicity we consider the 1D case. At excitation levels A near A_c $[\beta \equiv (|A_c - A|/A_c) \ll 1]$, small deviations $\Delta Y_j = Y_j(x,t) - Y_j^{(s)}$ of the parameters of the system from their values for the homogeneous state, $Y_j^{(s)}$, are described by expressions[4,5] $\Delta Y_j = a_j W(x,t)e^{ik_cx} + \text{c.c.} + O(\beta)$, where a_j are constants. The amplitude of these deviations satisfies[5,6] $W(x,t) \propto \beta^{1/2}$ and also satisfies a Ginzburg–Landau equation,[4–6]

$$\frac{\partial W}{\partial t} = -\gamma W + D\frac{\partial^2 W}{\partial x^2} + \alpha|W|^2W + \eta(x,t). \tag{2}$$

Here $\gamma \propto (A_c - A)A_c^{-1}$, $\alpha \sim 1$, and $D \propto k_c^{-2}$ are real constants; $\eta = \eta_R + i\eta_I$ is a random force; and the functions $\eta_R(x,t)$ and $\eta_I(x,t)$, with the selected $\xi_j(x,t)$ in (1), are δ-correlated and are of identical intensity $\Phi_\eta \lesssim \beta^{3/2} \ll 1$ (Ref. 5).

In real physical systems there are always small local inhomogeneities. They can be dealt with by adding terms $\lambda\varphi_j(x)$ to the right side of (1), where $\lambda \ll 1$. Assuming that the inhomogeneity is localized in the region $-\Lambda < x < \Lambda$, of size $2\Lambda \lesssim 2\pi/k_c$, we examine the distributions $\Delta Y_j(x,t)$ which arise in a slightly inhomogeneous system of this sort at excitation levels $A < A_c$.

We consider the interval of values of the parameter A in which we have $(A_c - A)As_c^{-1} \sim \lambda \ll 1$. We seek solutions ΔY_j in series form: $\Delta Y_j = \nu\Delta Y_j^{(0)} + \nu^2\Delta Y_j^{(1)} + \dots$, where $\nu^2 \equiv \lambda$. We substitute this expansion into (1) and go through the procedure[5] used to derive Eq. (2) for an ideally homogeneous system. We find that for the problem at hand, with a local inhomogeneity, the amplitude W of the $\Delta Y_j(x,t)$ distributions away from the inhomogeneity $(|x|\Lambda)$ is again described by Eq. (2), which is found from the equations of the first, second, and third approximations in the parameter ν. The boundary conditions on Eq. (2) at the boundaries of the inhomogeneous region, i.e., at the points $x = \pm\Lambda$, can be found by using the equations of the first and second approximations in ν:

$$D\frac{\partial W}{\partial x}\Big|_{x=-\Lambda} - D\frac{\partial W}{\partial x}\Big|_{x=\Lambda} = h, \quad W|_{x=\Lambda} = W|_{x=-\Lambda} + O(\nu^2), \tag{3}$$

where $h = \lambda\int_{-\Lambda}^{\Lambda}\exp(-ik_cx)\Sigma_{i=0}^{N}c_i\varphi_i(x)dx$, and c_i are constants.

In contrast with Refs. 4–6, we consider the case in which a subcritical bifurcation occurs in the homogeneous $(\lambda = 0)$ system at $A = A_c$ [correspondingly, we have a parameter value $\alpha > 0$ and $|\alpha| \sim 1$ in (2)]. In other words, large-amplitude structures form spontaneously at $A = A_c$. When this system contains a small inhomogeneity, problem (2), (3) in the absence of a noise $(\Phi_\eta = 0)$ has, under the condition $A < A_c$, two steady-state solutions which decay at infinity $(x \to \pm\infty)$:

$$W^{\pm}(x) = \sqrt{\frac{2\gamma}{\alpha}} \cosh^{-1} \left(\sqrt{\frac{\gamma}{D}}(|x| - \Lambda) + C^{\pm}\right) \exp(i\chi), |x| \geq \Lambda. \tag{4}$$

Here $\chi = \arg(h)$, and the constants $C^{\pm} (C^{+} < C^{-})$ are determined by the absolute value of the amplitude, $W_a^{\pm} = |W^{\pm}||_{x=\Lambda}$, which can be found, according to Eqs. (3) and (4), from the equation

$$\alpha(W_a^{\pm})^4 - 2\gamma(W_a^{\pm})^2 + \frac{1}{2}D|h|^2 = 0. \tag{5}$$

It follows from (5) that steady-state distributions $W^{\pm}(x)$ exist only at excitation levels A below a certain critical value, $A = A_c^- < A_c$. This critical value is found from the condition $\gamma^- \equiv \gamma(A_c^-) = \sqrt{(\alpha/2D)}|h| \sim \lambda$. At the point $A = A_c^-$, the solutions W^+ and W^- join $[W_0(x) = W^{\pm}(x)|_{A=A_c^-}]$, while at $A > A_c^-$ they disappear (Fig. 1a). A stability analysis of the steady-state distributions $W^{\pm}(x)$ shows that at $A < A_c^-$ one of them (W^-) is stable, and the other (W^+) unstable.

We consider the region $A < A_c^-$, where the state of the system is stable. Using the notation $\Delta_\gamma = \gamma - \gamma^-$, $\Delta W = (W - W_0(x)) \exp(-i_\chi)$, we can rewrite Eq. (2) as follows:

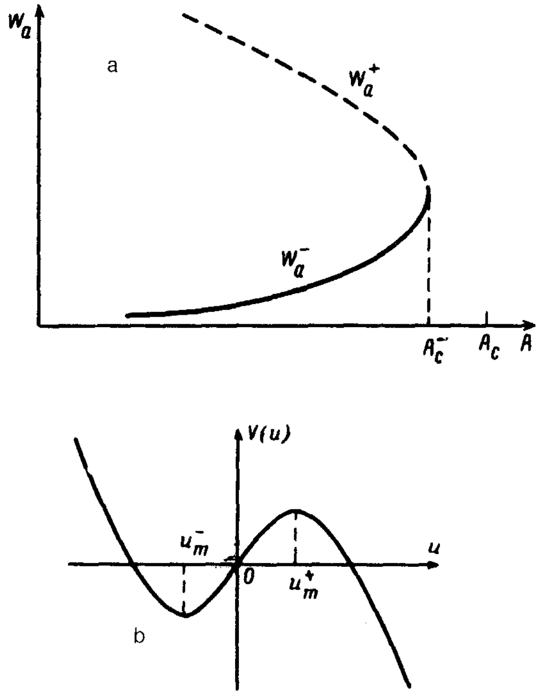

FIG. 1. a—Amplitude of the steady-state distributions of the parameters of the system near an inhomogeneity ($W_a = |W||_{x=\pm\Lambda}$) versus the excitation level A; b—the potential $V(u)$ at $A < A_c^-$.

$$\frac{\partial \Delta W}{\partial t} = -\hat{H}_0 \Delta W - \Delta\gamma(|W_0| + \Delta W) + \alpha[|W_0|^2(\Delta W^* - \Delta W)$$

$$+|W_0|(2|\Delta W|^2 + \Delta W^2) + |\Delta W|^2 \Delta W] + \exp(-i\chi)\eta(x,t), \quad (6)$$

where $\hat{H}_0 = -D(\partial^2/\partial x^2) + \gamma^- - 3\alpha|W_0(x)|^2$. Substituting $W^+(x)$ and $W^-(x)$ into (2), and subtracting one of the resulting expressions from the other, we find that the smallest eigenvalue of the operator $\hat{H}_0$ is zero and corresponds to an eigenfunction $\psi_0(x) = \sqrt{\alpha/2}\,\lim_{A \to A_c^-}(W^+(x) - W^-(x))\Delta\gamma^{-1/2}$.

To analyze Eq. (6), we assume that the noise intensity is low, $\Phi_\eta \ll \lambda^{3/2}$, and we assume $\Delta\gamma/\gamma^- \lesssim \mu^2 \ll 1$, where the small parameter μ satisfies the condition $\mu^3 \gtrsim \Phi_\eta \lambda^{-3/2}$. We write the solution of Eq. (6) in series form: $\Delta W = \mu \Delta W^{(0)} + \mu^2 \Delta W^{(1)} + \dots$. Transforming to the new variable $\tau = \mu t$, and expanding $\Delta W^{(0)}, \Delta W^{(1)}, \dots$ in the eigenfunctions of the operator $\hat{H}_0$, we find, in first order in μ, the result $\Delta W^{(0)} = u(\tau)\psi_0(x)$. In second order in μ we find from (6) an equation for $u(\tau)$:

$$\frac{du}{d\tau} = -\delta + bu^2 + \zeta(\tau), \quad (7)$$

where $\delta = g\Delta\gamma\mu^{-2}$; $g,b > 0$ are constants; and $\zeta(\tau)$ is a random force with a correlation function $\overline{\zeta(\tau_1)\zeta(\tau_2)} = (\Phi/\mu^3)\delta(\tau_1 - \tau_2)$, where $\Phi = \Phi_\eta(\int_{-\infty}^{+\infty} \psi_0^2(x)dx)^{-1}$.

Equation (7) describes the random motion of a "particle" in the potential $V(u) = u\delta - b/3\,u^3$ (Fig. 1b). At $A < A_c^-$, the potential $V(u)$ has two extrema, at the points $u_m^\pm = \pm(\delta/b)^{1/2}$, which correspond to respectively the unstable distribution W^+ and the stable one W^-. Using the solution of the problem of the average time (T) it takes a classical particle to escape from a potential well,[7] and working in the low-noise limit, $\Phi \ll (\Delta\gamma)^{3/2}$, we find an expression for the probability $P = T^{-1}$

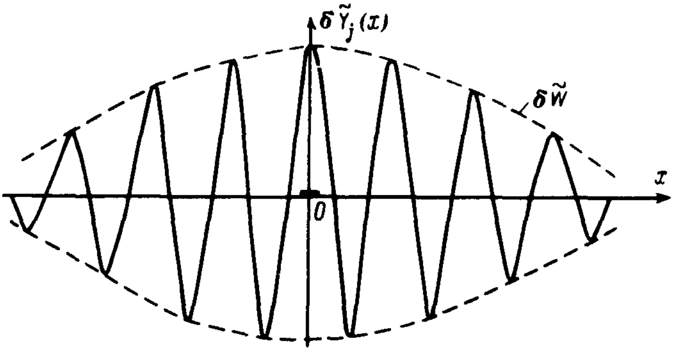

FIG. 2. Shape of the finite-amplitude critical fluctuation whose growth leads to the spontaneous appearance of an autosoliton.

that the quantity u will reach the region of values $u > u_m^+$ and will begin to grow without bound:

$$P = \frac{1}{\pi}(bg\Delta\gamma)^{1/2}\exp\left(-\frac{4}{3}\frac{(g\Delta\gamma)^{3/2}}{b^{1/2}\Phi}\right). \tag{8}$$

Expression (8) gives us the probability P (per unit time) that a local increase of an avalanche nature in the parameters of a stable ($A < A_c^-$; Fig. 1a) nonequilibrium system will occur near an inhomogeneity and will lead to the spontaneous formation of an autosoliton. The reason for the latter event is that a finite-amplitude critical fluctuation is localized in a spatial region of size $L \sim (D/\lambda)^{1/2}$ (Fig. 2). At the value $u = u_m^+$, the magnitude of the deviation from the stable steady state $W^-(x)$ is given by, according to (6) and (7), $\delta\widetilde{W} \equiv W - W^- = \mu(u_m^+ - u_m^-)\psi_0(x)e^{i\chi}$. We can thus find the shape of a critical fluctuation, $\delta Y_j = a_j \delta W \exp(ik_c x) + \text{c.c.}$:

$$\delta\widetilde{Y}_j(x) = 4a_j\left(\frac{g\Delta\gamma}{b}\right)^{1/2}\sinh(\tilde{x})\mathrm{ch}^{-2}(\tilde{x})\cos(k_c x + \chi), \quad |x| \geq \Lambda, \tag{9}$$

where $\tilde{x} = \sqrt{\gamma^-/D}(|x| - \Lambda) + 1 + \sqrt{2}$.

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Translated by D. Parsons