Charge relaxation in fractal structures

V. E. Arkhincheev
Buryatskiĭ Science Center, Siberian Branch of the Academy of Sciences of the USSR

(Submitted 6 August 1990)
Pis'ma Zh. Tekh. Fiz. **52**, No. 7, 1007–1009 (10 October 1990)

Charge relaxation in fractals can be described by a generalized equation in fractional time derivatives and is of a power-law nature.

In the present letter we report the results of an experimental study of charge relaxation in regular fractal structures. An equation describing charge relaxation in fractals is derived through renormalization-group transformations. This equation is written in terms of fractional time derivatives. The charge relaxation in fractals is only a power-law process, not an exponential Maxwellian process.

Charge relaxation in a conducting medium is described by the system of equations

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\vec{j} = 0, \quad \mathrm{div}\vec{e} = 4\pi\rho, \quad \vec{j} = \sigma\vec{e}. \tag{1}$$

Here ρ is the net charge density, $\bar{j}$ and $\bar{e}$ are the electric current and the electric field, respectively, and σ is the conductivity of the medium. (We assume that the medium has a unit permittivity.)

Let us perform some scale transformations on Eqs. (1). For this purpose we go over to the λ representation in time, and we write the spatial derivatives in the form of finite differences. For definiteness we consider Serpinskiĭ parquets (and also Koch curves). These structures are invariant under scale transformations 2^n, where $n = 1, 2, 3, \ldots$. (The very simple triangular Serpinskiĭ parquet is found in the following manner: An equilateral triangle is divided into four equal triangles; the central triangle is discarded, and the same procedure is carried out for each of the remaining parts; and so on, *ad infinitum*.[1]) We fix a certain scale A. The next step is to make the transition from scale A to the large scale $2A$. Direct calculations analogous to those of Ref. 2 easily verify that the parameter λ increases by a factor of 5, and the particle density by a factor of 3, in the low-frequency limit. We thus obtain a renormalized equation describing charge relaxation in fractals:

$$5\lambda\rho(\lambda, x) = -3(4\pi\sigma)\rho(\lambda, x). \tag{2}$$

The invariance of this equation under scale transformations implies a power-law frequency dependence:

$$\lambda^S \rho(\lambda, x) = -a\rho(\lambda, x), \tag{3}$$

where $S = \ln 5/\ln 3$, and α is a constant. (Corresponding calculations for a Koch curve yield an index $S = 1/2$.) Using the definition $(\partial^S f/\partial t^S) = \lambda^S f_\lambda$, of a fractional derivative, we find the equation which we have been seeking, for charge relaxation in fractals in the t representation

$$\frac{\partial^S \rho}{\partial t^S} + a\rho = 0. \tag{4}$$

The Green's function of Eq. (4) is

$$G_\lambda = \frac{1}{\lambda^S + a}. \tag{5}$$

In general, the form of this equation is determined by the method of steepest descent. For the index $S = 1/2$ the Green's function can be calculated explicitly. Using the identity

$$\int_0^\infty \exp(-a\tau)d\tau = 1/a,$$

we find the Green's function in the t representation:

$$G(t) = \int_0^\infty \int \exp(-\sqrt{\lambda}\tau - a\tau + \lambda t)\frac{d\lambda}{2\pi}d\tau = \int_0^\infty \exp(-\frac{\tau^2}{4t} - a\tau)\frac{\tau}{2t}\frac{d\tau}{\sqrt{\pi t}} . \tag{6}$$

The charge relaxation in fractals is thus of a power-law nature for long segments of time. The power-law behavior of the charge relaxation occurs because fractal structures do not have characteristic dimensions. In other words, there can be metallic inclusions of all sizes in which the charge will relax.

Charge relaxation in percolation systems at the percolation threshold is also described by an equation like (4). For the three-dimensional case, this point can be verified easily by making use of a scaling hypothesis for the effective conductivity with respect to frequency.[3] (A possible value of the index for the three-dimensional case, $S = 1/3$, was pointed out in Ref. 4.) In the two-dimensional case, this assertion follows from the results of Ref. 5.

[1] B. Mandelbrot, *Fractal Geometry of Nature*, San Francisco, 1982.
[2] R. Rammal and G. Toulouse, J. Phys. Lett. (Paris) b44, L13 (1983).
[3] B. I. Shklovskiĭ and A. L. Éfros, Phys. Status Solidi **76**, 475 (1976).
[4] V. E. Arkhincheev, Pis'ma Zh. Eksp. Teor. Fiz. **50**, 293 (1989) [JETP Lett. **50**, 325 (1989)].
[5] V. E. Arkhincheev, Zh. Eksp. Teor. Fiz. **97**, 1379 (1990) [Sov. Phys. JETP **70**, 776, (1990)].

Translated by D. Parsons