Twistor displacement in the equations of relativistic particles and strings

D. V. Volkov, V. A. Soroka, D. P. Sorokin, and V. I. Tkach
Kharkov Physicotechnical Institute, Academy of Sciences of the Ukrainian SSR

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A generalization of the equations of motion of relativistic massless particles in external fields and relativistic strings is studied. The generalization is based on the introduction of the twistor displacement operation and contains a fundamental constant with the dimensions of length.

Beginning with the work of Penrose,[1] twistors have been widely used for describing massless relativistic particles and superparticles, and also strings and superstrings. As an alternative to the usual coordinate description, the twistor approach in a number of cases gives a more economical description of the constraints and a clearer representation of the symmetry properties. Nevertheless, no important modifications of the current theory attributable to twistors have, to the best of our knowledge, been suggested.

In the present article we would like to show that starting from the twistor formulation of the dynamics of massless particles and strings it is possible to generalize the corresponding Lagrangians such that a certain fundamental length appears in the

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theory. The application of this generalization to the case of (super)strings is undoubtedly the most interesting from the physical point of view, but the basic idea can be illustrated by the simple example of a relativistic massless particle in spacetime of dimension $D = 1 + 2$.

As is well known,[2] the dynamics of such a particle can be described by a Lagrangian of the form

$$L_0 = \lambda_\alpha \lambda_\beta \dot{x}^{\alpha\beta}, \tag{1}$$

which acts as a bridge between the space-time and the twistor formulations[3,4] [λ_α is a commuting Majorana spinor, $\alpha, \beta = 1,2$, $x^{\alpha\beta} = (1/\sqrt{2})\gamma^{m\alpha\beta} x_m(\tau)$ is the particle coordinate ($m = 0,1,2$), and $\dot{x}^{\alpha\beta} = (d/d\tau)x^{\alpha\beta}(\tau)$ is the particle velocity].

The mass shell condition $p_m p^m = 0$ for a particle with $m = 0$ arises from (1) as a consequence of the constraint

$$p_{\alpha\beta} = \lambda_\alpha \lambda_\beta \tag{2}$$

and the identity $\lambda_\alpha \lambda_\beta \epsilon^{\alpha\beta} = \lambda_\alpha \lambda^\alpha = 0 (\epsilon^{12} = -\epsilon^{12} = 1)$.

We assume that the particle interacts with certain quantum fields whose vacuum fluctuations can, in general, lead to additional terms containing $\dot{\lambda}_\alpha$ in the Lagrangian (1); then the minimal generalization of (1) has the form

$$L = L_0 + L_1 = L_0 + l\lambda_\alpha \dot{\lambda}_\beta \epsilon^{\alpha\beta}, \tag{3}$$

where l is an arbitrary parameter with the dimensions of length. The dynamics of a particle described by the Lagrangian (3) is constrained by the relations (2) and

$$\varphi_\alpha = \pi_\alpha + l\lambda_\alpha = 0 \tag{4}$$

(π_α is the momentum which is canonically conjugate to λ^α), one of which, $p_{\alpha\beta}\lambda^\alpha \lambda^\beta = 0$, is a first-class constraint corresponding to the local reparametrization invariance of the theory, while the other four are second-class constraints. Consequently, the Lagrangians (1) and (3) have the same number of independent canonical variables, and the λ_α are auxiliary variables. The spinor constraints (4) satisfy the Poisson brackets $[\varphi_\alpha, \varphi_\beta] = -2l\epsilon_{\alpha\beta}$.

In order to isolate from (2) the two remaining second-class constraints, we introduce the spinor $\mu^\alpha = x^{\alpha\beta}\lambda_\beta$ (which together with λ^β forms a twistor) (Refs. 3 and 4) and we project (2) into (4) onto λ_α, μ^β:

$$\Phi^1 = p_{\alpha\beta}\mu^\alpha \mu^\beta - (\lambda\mu)^2 - \frac{1}{l}(\mu_\alpha \varphi^\alpha)(\lambda\mu) = 0, \quad \lambda\mu \equiv \lambda_\alpha \mu^\alpha,$$

$$\Phi^2 = p_{\alpha\beta}\lambda^\alpha \mu^\beta - \frac{1}{2l}(\lambda_\alpha \varphi^\alpha)(\lambda\mu) = 0. \tag{5}$$

The Poisson brackets of the constraints (5) have the form

$$[\Phi^i, \Phi^k] = \frac{1}{l}(\lambda\mu)^3 \epsilon^{ik} \quad (i, k = 1, 2). \tag{6}$$

The transformation to Dirac brackets

$$[f,g]^* = [f,g] - \frac{1}{2l}[f,\varphi^\alpha]\epsilon_{\alpha\beta}[\varphi^\beta,g] - \frac{l}{(\lambda\mu)^3}[f,\Phi^i]\epsilon_{ik}[\Phi^k,g] \tag{7}$$

causes the $x^{\alpha\beta}$ to no longer commute[1)]

$$[x^{\alpha\beta},x^{\gamma\delta}]^* = \frac{l}{2(\lambda\mu)^2}(\epsilon^{\alpha\delta}\mu^\gamma\mu^\beta + \epsilon^{\beta\gamma}\mu^\alpha\mu^\delta), \tag{8}$$

and the commutation relations between λ^α and μ^β take the form

$$[\lambda^\alpha,\lambda^\beta]^* = 0, \quad [\mu^\alpha,\lambda^\beta]^* = -\frac{1}{2}\epsilon^{\alpha\beta}, \tag{9a}$$

$$[\mu^\alpha,\mu^\beta]^* = \frac{l}{2}\epsilon^{\alpha\beta}. \tag{9b}$$

In order to restore the canonical nature of the Dirac brackets for the twistor variables λ, μ, we make a twistor displacement μ^d in the direction of λ^α

$$\hat{\mu}^\alpha = \mu^\alpha + \frac{l}{2}\lambda^\alpha, \tag{10}$$

after which the brackets (9b) for $\hat{\mu}$ vanish. Assuming that $\hat{\mu}^\alpha$ is a function of λ_α and the new variable $\hat{x}^{\alpha\beta}$($\hat{\mu}^\alpha = \hat{x}^{\alpha\beta}\lambda_\beta$), we find the relation between $\hat{x}^{\alpha\beta}$ and the old coordinates $x^{\alpha\beta}$:

$$\hat{x}^{\alpha\beta} = x^{\alpha\beta} + \frac{l}{2(\lambda\mu)}(\lambda^\alpha\mu^\beta + \lambda^\beta\mu^\alpha). \tag{11}$$

By virtue of the constraints (2) and the definition of μ^α, the twistor displacement $x^{\alpha\beta}$ (11) is proportional to the orbital angular momentum. It is easy to verify that the $\hat{x}$ commute in brackets of the form in (7), while the Lagrangian (3), after the redefinition (11), becomes the Lagrangian (1), which indicates that the corresponding free particle theories are equivalent. Therefore, in the absence of an interaction it is impossible to tell which of the coordinates $x^{\alpha\beta}$ or $\hat{x}^{\alpha\beta}$ are the physical coordinates of the particle. The answer to this question can be obtained only by including the interaction with external fields.

Let us consider, for example, the dynamics of a particle interacting minimally with the electromagnetic field $A_{\alpha\beta}(x)$:

$$L_B = -\dot{x}^{\alpha\beta}A_{\alpha\beta}(x). \tag{12}$$

The constraint structure of the theory remains unchanged, with the only difference that in (2) and (5) $p_{\alpha\beta}$ are replaced by the covariant momenta $D_{\alpha\beta} = p_{\alpha\beta} + A_{\alpha\beta}$, while the right-hand side of (6) involves the electromagnetic field strength tensor $F_{mn} = \partial_{[m}A_{n]}$, which leads to a modification of the Dirac brackets.

As in the free case, we can use the twistor displacement (10), (11) to transform to the representation in which the $\hat{x}$ commute. Here an infinite series in l^n of nominal interaction terms containing F_{mn} and its derivatives arises in the Lagrangian. In first order in l this Lagrangian can be written as

$$L = \lambda_\alpha \lambda_\beta \dot{\hat{x}}^{\alpha\beta} - \dot{\hat{x}}^{\alpha\beta}\left(A_{\alpha\beta} + \frac{l}{2}F_{\alpha\beta}\right), \tag{13}$$

where

$$F_{\alpha\beta} = \frac{1}{\sqrt{2}}\gamma^l_{\alpha\beta}\epsilon_{lmn}F^{mn}.$$

The above discussion can be generalized to the case of higher dimensions $D = 1 + 3,\ 1 + 5,\ 1 + 9$.

We note that the presence of the additional term L_1 must, in general, also affect the equations of motion of the external fields, which should be manifested both in the self-consistency conditions when considering superparticles interacting with external fields, and in string and superstring theories.

In conclusion, we give the generalization of the action for classical strings to dimension $D = 1 + 3$:

$$S = \int d\tau d\sigma \sqrt{-g}\{\bar{\lambda}^{\dot{A}}\rho^\mu \lambda^A(\partial_\mu x_{A\dot{A}} - \frac{L^2}{2}\bar{\lambda}_{\dot{A}}\rho_\mu \lambda_A) + \frac{l}{2}(\lambda^A \rho^\mu \partial_\mu \lambda^B \epsilon_{AB} + \text{H.c.})\}, \tag{14}$$

where ρ^μ are the Dirac matrices on the world sheet, λ_A, $\bar{\lambda}_{\dot{A}}$ are spinors in both $D = 1 + 3$ and $d = 1 + 1$, and L is the string length parameter. The first term in (14) is equivalent to the action of the classical bosonic string, and the second is the generalization of L_1 in (3), where both terms are reparametrization and scale invariant.

A detailed investigation of the action (14) will be carried out in a separate study.

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[1)] For the gauge choice $x^0 = \tau$ the Dirac brackets for the coordinates have the simple form $[x_1, x_2]^* = -l/\sqrt{2}E$, where E is the particle energy.

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