Vertex operator representation of the Weyl-Moyal-Fairlie sine algebra

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An irreducible representation of the Weyl-Moyal-Fairlie sine algebra in terms of vertex operators is constructed. The embedding of the sine algebra in $\widehat{gl}(\infty)$ is described.

1. The Weyl-Moyal-Fairlie sine algebra g is a known quantum deformation of the Lie algebra of Hamiltonian vector fields on the torus with a central extension. In the Fairlie realization it is specified by the generators T_{n}, c and the commutation relations

$$[T_{\vec{n}}, T_{\vec{m}}] = 2i \sin \hbar(\vec{n} \times \vec{m}) T_{\vec{n}+\vec{m}} + n_1 \delta_{\vec{n}+\vec{m}} c, \tag{1}$$

where $\mathbf{n}$ and $\mathbf{m}$ are integer vectors of the two-dimensional plane. The algebra (1) is an example of a wide class of continuous contragradient Lie algebras introduced in Ref. 2. Applications of this type of algebra to the theory of integrable equations (nonlocal analogs of two-dimensional Toda chains) are also described in that study. Algebras of the type (1) also arise as an intermediate step in the theory of the deformation of integrable Korteweg-deVries equations into integrable integrodifferential equations of the TLW type.[3] There are indications that algebra (1) is important in the nonperturbative formulation of string theory.[4]

In this article we describe the construction of an irreducible representation of highest weight of algebra (1) in terms of vertex operators. We would like to draw special attention to the fact that calculation of the rules for combining vertex operators and commutation relations gives rise to pole separation effects which are similar to those that appear in Refs. 6 and 7 in the construction of the basis representation of Kac-Moody algebras quantized in the manner of Drinfeld and Jimbo. It seems to us that this is not accidental. This is suggested by the fact that for $c = 0$ the relations (1) follow from the equations $T_{\mathbf{n}} T_{\mathbf{m}} = e^{i\hbar(\mathbf{n}\times\mathbf{m})} T_{\mathbf{n}+\mathbf{m}}$ or $T_{\mathbf{n}} T_{\mathbf{m}} = e^{2i\hbar(\mathbf{n}\times\mathbf{m})} T_{\mathbf{m}} T_{\mathbf{n}}$. For $\mathbf{n} = (1,0)$, $\mathbf{m} = (0,1)$ these are relations that determine the quantum surface. The quantum group $GL_q(2)$ is the group of automorphisms of the quantum surface.[8] Therefore, it is reasonable to expect that there is a relationship the representation of the quantum two-loop algebra (1) that we have constructed and quantum group representations.

2. The Lie algebra g contains an infinite number of infinite-dimensional Heisenberg subalgebras $s^{(l)} = \{T_{n,ln} \,|\, n \in \mathbb{Z}\setminus 0\}$, where l takes all integer values $\mathbb{Z}$. The maximal commuting subalgebra is the algebra $H = \{T_{0,n} \,|\, n \in \mathbb{Z}\}$. We fix some l and write $X_k(z) = \Sigma_{n\in Z} T_{n,nl+k} z^{-n}$, where $k \in \mathbb{Z}\setminus 0$ and z is a complex variable. All the generators of the algebra g are contained in $s^{(l)}$ and the fields $X_k(z)$. It is easily verified that the following relations are satisfied:

$$[T_{n,nl},\ X_k(z)] = 2i\sin(\hbar nk) z^n X_k(z),$$

$$[T_{-n,-nl},\ X_k(z)] = -2i\sin(\hbar nk) z^{-n} X_k(z). \tag{2}$$

The algebra $s^{(l)}$ has the standard irreducible representation in the space of polynomials of an infinite number of variables $V = \mathbb{C}[x_1, x_2, \ldots]$

$$\pi_0(T_{n,nl}) = \partial/\partial x_n,\quad \pi_0(T_{-n,-nl}) = n x_n,\quad \pi_0(c) = 1.$$

Equations (2) with $T_{\pm n, \pm nl}$ represented by the generators $\partial/\partial x_n$ and $n x_n$ have a unique solution (up to multiplication by constants c_k) in the class of differential operators on the space $\mathbb{C}[x_1, x_2, \ldots]$ (Ref. 5):

$$\hat{X}_k(z) = c_k e^{2i\sum_{m\geq 1} z^m \sin(\hbar k m) x_m}\, e^{2i\sum_{m\geq 1} \frac{z^{-m}}{m} \sin(\hbar k m)\, \partial/\partial x_m}. \tag{3}$$

We introduce the generators $\hat{X}_{k,n} = 1/2\pi i \oint_c dz z^{n-1} \hat{X}_k(z)$, where the integration runs over a contour c enclosing the point 0. Our result is the proof that the correspondence

$$\pi(T_{n,nl+k}) = \hat{X}_{k,n},\quad k \neq 0$$

$$\pi(T_{n,nl}) = \partial/\partial x_n;\quad \pi(T_{-n,-nl}) = n x_n;\quad \pi(c) = 1 \tag{4}$$

specifies the irreducible representation of highest weight $\Lambda \in H^*$, where $\Lambda(T_{0,k}) = c_k$, in the space V with vacuum vector $|0\rangle = 1$.

3. The proof consists of the direct verification of the commutation relations between the generators $\hat{X}_{k,n}$. We write the rule for combining vertex operators as

$$\hat{X}_k(z)\hat{X}_{k'}(\varsigma) = \frac{(z - q^{k-k'}\varsigma)(z - q^{-(k-k')}\varsigma)}{(z - q^{k+k'}\varsigma)(z - q^{-(k+k')}\varsigma)} : X_k(z)X_{k'}(\varsigma) :, \tag{5}$$

where $|z| > |\zeta|$ and $q = e^{i\hbar}$. The normal ordering is understood in the standard sense. Since the fraction on the right side of (5) is not changed by the substitution $z \leftrightarrow \zeta$, $k \leftrightarrow k'$, the calculation of the commutators $[\hat{X}_{k,n}\hat{X}_{k',m}]$ for $k + k' \neq 0$ reduces the calculation of the contour integrals

$$\sum_{i=1}^{2} \frac{1}{(2\pi i)^2} \oint_c d\varsigma \varsigma^{n-1} \oint_{\Gamma_i} dz z^{n-1}$$

of the right side of (5). Here the Γ_i are infinitesimal circles enclosing the poles of (5) $z = q^{k+k'}\zeta$ and $z = q^{-(k+k')}\zeta$, respectively. In the case where $k + k' = 0$, two first-order poles into a single second-order pole and we have the standard case. The result of the calculations is the following. When the constants c_k satisfy the functional equation

$$c_k c_{k'} = \frac{q^{k+k'} - q^{-(k+k')}}{(q^k - q^{-k})(q^{k'} - q^{-k'})} c_{k+k'}, \quad q = e^{i\hbar}, \tag{6}$$

whose solution $c_k = q^k/(q^k - q^{-k})$ is unique up to multiplication by a phase factor $e^{ik\lambda}$, the commutation relations are

$$[\hat{X}_{k,n}, \hat{X}_{k',m}] = 2i \sin \hbar(nk' - mk)\hat{X}_{k+k',n+m}, \text{for} \quad k + k' \neq 0$$

$$[\hat{X}_{k,n}, \hat{X}_{-k,n}] = \begin{cases} -2i \sin(\hbar kp)\partial/\partial x_p, & p = n + m > 0 \\ 2i \sin(\hbar kp)px_p, & p = -(n+m) > 0\,. \\ n \cdot 1, & n + m = 0 \end{cases} \tag{7}$$

Comparison of (7), (4), and (1) shows that the algebra of the differential operators $\hat{X}_{k,n}$, $\partial/\partial x_m$, x_m, 1, $k \in \mathbb{Z} \setminus 0$, $m > 0$ closes (which is not obvious *a priori*) and specifies a representation of the Lie algebra (1) in the space V. Space the restriction of the representation π to the Heisenberg subalgebra $s^{(l)}$ is irreducible, π is also irreducible. From the explicit form of $\hat{X}_k(z)$ it is easy to see that $\hat{X}_{k,0}|0\rangle = c_k$ and $X_{k,n}|0\rangle = 0$ for $n > 0$. This means that the constructed representation is the representation of highest weight Λ under the decomposition $g = N_+ + H + N_-$, where N_+ (N_-) is generated by the generators $T_{n,k}$, $n > 0$ ($n < 0$), and H is generated by the generators $T_{0,k}$ and is the maximal commuting subalgebra in g. The weight $\Lambda \in H^*$ is specified by its values on the basis H by the conditions $\Lambda(T_{0,k}) = c_k$.

It is easy to prove that representations corresponding to different choices of Heisenberg subalgebras are equivalent.

4. It is not difficult to verify that the vertex operators $\hat{X}_k(z)$ (3) are obtained from the standard vertex operator $Z(p,q)$ (Ref. 9) which realizes the basic representation of $\widehat{gl}(\infty)$ by means of the reduction $p = ze^{ik\hbar}$, $q = ze^{-ik\hbar}$. We thus derive the equation specifying the embedding of the sine algebra in $\widehat{gl}(\infty)$:

$$T_{n,nl+k} = e^{i\hbar kn} \sum_{m\in\mathbb{Z}} :\psi_m \psi^*_{m+n}: e^{2i\hbar km} + \delta_{n,0} \frac{e^{i\hbar k}}{e^{i\hbar k} - e^{-i\hbar k}}, \tag{8}$$

where ψ_m, ψ^*_m, $m\in\mathbb{Z}$ are the generators of the Clifford algebra (free fermions) with the defining relations $\{\psi_i, \psi_j\} = \{\psi^*_i,\psi^*_j\} = 0$ $\{\psi_i,\psi^*_j\} = \delta_{ij}$. The fermionic normal ordering is done in the usual way: $:\psi_i\psi^*_j: = \psi_i\psi^*_j - \langle\psi_i,\psi^*_j\rangle$, where $\langle\psi_i,\psi^*_j\rangle = Y_-(i)\delta_{i,j}$; $Y_-(i) = 1$ if $i\leqslant 0$, and $Y_-(i) = 0$ if $i>0$.

5. It is instructive to trace the application of this construction in the limiting case $\hbar\to 0$, which after renormalization of the generators corresponds to the central extension of the algebra of Hamiltonian vector fields on the torus with the commutation relations

$$[T_{\vec{n}}, T_{\vec{m}}] = [\vec{n}\times\vec{m}]T_{\vec{n}+\vec{m}} + n_1\delta_{\vec{n}+\vec{m}}c.$$

Equations (2) and (3) are preserved, but sin $\hbar mk$ is replaced by mk, which causes an essential singularity to appear in (5), so that the corresponding algebra of the operators $\hat{X}_{k,n}$, $\partial/\partial x_m$, x_m, 1 does not close. Therefore, the construction we have described is not applicable in this case.

Since our construction gives, by construction, special values of the central charge, it would be interesting to apply the construction of Ref. 10 to the construction of a representation of the algebra (1) with arbitrary central charge.

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