General deformations of coset models and Grassmann manifolds

A. A. Belov and Yu. E. Lozovik
Institute of Spectroscopy, USSR Academy of Sciences

(Submitted 18 October 1990)
JETP Lett. **52**, No. 10, 1170–1172 (25 November 1990)

For a class of conformal field theories within the framework of the affine-Virasoro construction it is shown that the deformation manifold is a Grassmann manifold.

The general unitary affine-Virasoro construction, recently proposed by Morozov *et al.*[1] and independently by Halpern *et al.*,[2] which includes solutions of the Sugawara type, parafermions, and coset models as limiting cases, can be used to obtain a series of conformal field theories possessing nontrivial deformations for fixed central charge $c \geqslant 1$.

In the present article we construct new series of conformal theories and show that the "moduli space," which describe the deformations of these theories, is a Grassmann manifold.

The deformable conformal theories (for constant c) presently known reduce either to regular "quasi-Abelian" case (Cartan deformations, the case $k=1$, and the high level limit $k \to \infty$; Ref. 2), or arise sporadically, for example, $SU(2)_4^{\#}$ (Ref. 1), and seem rather mysterious. We propose a regular, although not exhaustive, method of constructing deformable theories $F^{\#}_{k_0}$ which are a generalization of $SU(2)^{\#}_4$.

1. Let us begin with the illustrative example of the coset model $M_k(n;m,n-m) = SO(n)_k/SO(n-m)_k \times SO(m)_k$. We embed the model into the diagonal ansatz

$$L = \sum_{A<B}^{n} S_{AB} : J_{AB}(z) J_{AB}(z) :, \tag{1}$$

where $J_{AB}(z)$ are the currents of the KM algebra $SO(n)_k$, and we show that all coset models $M_2(n;m,n-m)$ with given n and $m \in \{2,...,n-1\}$ are related to each other by marginal deformations. Substituting into the Virasoro conditions

$$\frac{1}{2} S_{AB} = \sum_{c}^{(AB)} [S_{AB}(S_{AC} + S_{BC}) - S_{AC}S_{BC}] + kS_{AB}^2 \tag{2}$$

the "inertia tensor" of the easily deformed coset solution

$$S_{AB} = S_{AB}^{(0)} + X_{AB},$$

where

$$S_{\alpha\beta}^{(0)} = \frac{1}{2(n+k-2)} - \frac{1}{2(m+k-2)}, \qquad \alpha, \beta = 1, ..., m;$$

$$S_{ab}^{(0)} = \frac{1}{2(n+k-2)} - \frac{1}{2(n-m+k-2)}, \qquad a, b = m+1, ..., n$$

and

$$S_{\alpha a}^{(0)} = \frac{1}{2(n+k-2)},$$

we discover that for $k=2$ (and only in that case) in the approximation linear in X_{AB} the corank of the system is $n-2$. (The "free" parameters have the form $X_{\alpha a} - X_{\beta a}$ and $X_{\alpha a} - X_{\alpha b}$). It can be verified that the inclusion of higher-order corrections in X_{AB} preserves this "degeneracy." We shall show that a global solution exists. The substitution

$$S_{AB} = -\frac{R^A R^B}{2} \tag{3}$$

reduces (2) to the system of equations

$$\sum_{A=1}^{n} R^A = 0, \qquad \sum_{A=1}^{n} (R^A)^2 = 1, \tag{4}$$

defining the deformation manifold $\mathbb{R}P^{n-2}$. Coset models $M_2(n;m,n-m)$correspond to the point $R^\alpha (n-m)/nm)^{1/2}$, $\alpha = 1,...,m$, $R^a = -[m/n(n-m)]^{1/2}$, $a = m+1,...,n$. Therefore, all the coset models in question "lie" on $\mathbb{R}P^{n-2}$, and therefore are related to each other by marginal deformations.

2. Let us now turn to the more general case of coset models for the form

$$M_k(n, \{m_i\}_{i=1}^p) = SO(n)_k|\Pi_{i=1}^p SO(m_i)_k, \quad \sum_{i=1}^p m_i = n.$$

The central charge for them has the value $c = p - 1$. We embed these models in the diagonal ansatz. For $k = 2$ we find that in the vicinity of the coset models there exist $(p-1)(n-p)$-parameter deformations. Substitution of the ansatz

$$S_{AB} = -\frac{1}{2}\sum_{i=1}^{q} R_i^A R_i^B, \tag{5}$$

which is a direct generalization of (3), into (2) leads to the system of equations

$$\sum_{A=1}^{n} R_i^A = 0, \qquad i = 1, \dots, q,$$

$$\sum_{A=1}^{n} R_i^A R_j^A = \delta_{ij}, \qquad i < j, \tag{6}$$

which determines the Schtifel manifold $V_q(\mathbb{R}^{n-1})$. Taking into consideration the fact that the q-reference vector R_i^A is determined up to an orthogonal transformation from $O(q)$, we finally find that the moduli space of deformable conformal theories coincides with the Grassmann manifold $G_q(\mathbb{R}^{n-1})$. Here a new geometrical interpretation of the central charge c arises: it is equal to the dimension of the reference vector q. The manifold of the deformations of all $SO(n_2)$ coset models with given central charge c is an infinite-dimensional Grassmann manifold $G_c(\mathbb{R}^\infty)$ with the natural topology of the inductive limit.

3. In the construction of the $SO(n)_2^{\#}$-series of deformable models we have implicitly used the embedding $SO(n)_2 \subset SU(n)_1$. The formalization and generalization of this procedure allow the construction of new series of deformable theories. Let us consider, for example, the case $SO(n)_2 \oplus SO(n)_2 \subset SO(2n)_1$. The corresponding energy-momentum tensor has the form

$$L = \sum_{K<L} M_{KL} : (J_{KL} + \tilde{J}_{KL})^2 : + \sum_{K<L} L_{KL} : (J_{KL} - \tilde{J}_{KL})^2 : . \tag{7}$$

We note that ansatz (7) no longer has the usual diagonal form. For M_{KL} and L_{KL} the Virasoro conditions have the form

$$\frac{1}{2}L_{KL} = kL_{KL}^2 + \sum_{m}^{(KL)} \{L_{KL}(M_{KM} + M_{LM} + L_{KM} + L_{LM}) - L_{KM}L_{LM} - M_{KM}M_{LM}\},$$

$$\frac{1}{2}M_{KL} = kM_{KL}^2 + \sum_{m}^{(KL)} \{M_{KL}(M_{KM} + M_{LM} + L_{KM} + L_{LM}) - L_{KM}M_{LM} - L_{LM}M_{KM}\}. \tag{8}$$

In addition to the solutions $L_{KL} = M_{KL}$ reducing to (5), the system (8) has new nontrivial solutions which describe multiparameter deformations. As before, the deformation spaces are Grassmann manifolds.

It would be interesting to study polymodal deformations of conformal theories from the viewpoint of catastrophe theory in view of the parallels between their classifications observed in Ref. 3.

[1]A. Yu. Morozov *et al.*, Int. J. Mod. Phys. **A5**, 803 (1990).
[2]M. B. Halpern and E. Kiritsis, Mod. Phys. Lett. **A4**, 1373 (1989); M. B. Halpern *et al.*, Int. J. Mod. Phys. **A5**, 2275 (1990).
[3]C. Vafa and N. Warner, Phys. Lett. **218B**, 51 (1989).

Translated by Patricia Millard