Generalization of the Gamow formula to the multidimensional case

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(Submitted 3 April 1991)
Pis'ma Zh. Eksp. Teor. Fiz. **53**, No. 9, 433–436 (10 May 1991)

The Gamow formula for a level width is generalized to the case of multidimensional systems with separable variables. A condition for the applicability of this approximation is found.

1. The problem of calculating the energies and widths of quasistationary states (resonances) arises frequently in atomic and nuclear physics. If the potential $U(x)$ is smooth, and the width Γ small, the Gamow formula[1,2] can be used in the one-dimensional case. In the present letter we consider a generalization of the formula to multidimensional systems with separable variables $q_1, q_2, \ldots, q_f$, where f is the number of degrees of freedom.

2. If the barrier transmission is exponentially low, the tunneling in a multidimensional potential occurs essentially along one of the coordinates. We choose[1)] this coordinate to be q_f. Modifying the Bohr–Sommerfeld quantization rule to incorporate the transmission of the barrier,[3,4] we find the formula we are seeking:

$$\Gamma = cT_f^{-1}\exp(-2\pi a_f), \tag{1}$$

where

$$c = \left(\alpha \bar{v}_f \sum_{i=1}^{f} 1/\bar{v}_i\right)^{-1}, \quad T_f = 2\int_{q_f^{(0)}}^{q_f^{(1)}} p_f^{-1} dq_f \tag{1a}$$

$$a_i = \frac{1}{\pi}\mathrm{Im}S_i = \frac{1}{\pi}\int_{q_i^{(1)}}^{q_i^{(2)}} (-p_i^2)^{1/2} dq_i, \tag{1b}$$

$$S = \sum_{i=1}^{f} S_i = \sum_{i=1}^{f} \int p_i dq_i, \quad p_i = \{2[\alpha E - u_i - \beta_i v_i]\}^{1/2}. \tag{2}$$

Here S is the action, E is the energy, β_i are separation constants ($\Sigma_{i=1}^{f}\beta_i = \text{const}$), α is a constant which is determined in the course of the separation of variables,[2] T_f is the period of the oscillation of the classical particle along q_f, $q_i^{(k)}$ are the turning points ($k = 0, 1, 2$), and the superior bar means the expectation value calculated from the semiclassical wave function:

$$\bar{v}_i = \int v_i(q)\psi^2(q)dq \approx \int_{q_i^{(0)}}^{q_i^{(1)}} \frac{v_i(q)}{p_i(q)}dq \Big/ \int_{q_i^{(0)}}^{q_i^{(1)}} \frac{dq}{p_i(q)}. \tag{3}$$

Here $q_i^{(0)} < q_i < q_i^{(1)}$ is the classically allowed region of motion along the coordinate q_i, and $q_i^{(1)} < q_i < q_i^{(2)}$ is the tunneling region, in which we have $p_i^2 < 0$ (Fig. 1 in Ref. 4). The distinction between the multidimensional problem and the one-dimensional problem is in the coefficient of the exponential function c, which effectively incorporates the influence of the motion along the coordinates $q_i (i \neq f)$ on the number of times a particle collides with the barrier wall at $q_f = q_f^{(1)}$.

3. Equation (1) reduces the calculation of the width Γ to quadrature form. This equation could have a variety of applications. We will illustrate its use here in the particular example of the Stark effect in the hydrogen atom.

Quasistationary states in a uniform electric field $\mathscr{E}$ are characterized by parabolic quantum numbers n_1, n_2, m (below we use $m \geqslant 0$; $n = n_1 + n_2 + m + 1$ is the principal quantum number of the level). The quantities $E = E_r - i\Gamma/2$ and $\beta_{1,2}$ are found as functions of the field $\mathscr{E}$ from the quantization conditions[2,5] in terms of the variables ξ and η. In the case at hand we have $f = 2$ and

$$u_i(q) = \frac{1}{8}[m^2q^{-2} - (-1)^i\mathscr{E}q], \quad v_i(q) = -\frac{1}{2q}, \quad \alpha = 1/4, \quad \beta_1 + \beta_2 = 1, \tag{4}$$

where $q = \xi, \eta$ for $i = 1, 2$. Under the assumption that the tunneling of the electron occurs along the coordinate η and that the effective potential along ξ is a retarding potential, we find from (1)

$$\Gamma^{(n_1n_2m)}(\mathscr{E}) = \frac{4}{(1+\gamma)T_\eta}\exp(-2\pi a_\eta). \tag{5}$$

Here

$$a_\eta = \frac{n(-\epsilon)^{3/2}}{2\pi F}\int_{t_1}^{t_2}\frac{dt}{t}(A - Bt + t^2 - t^3)^{1/2}, \tag{6}$$

$$T_\eta = 4n^3\int_{y_0}^{y_1} dy(\epsilon - \mu^2y^{-2} + 4\beta_2y^{-1} + Fy)^{1/2},$$

$A = \mu^2F^2/(-\epsilon)^3$, $B = 4\beta_2F/\epsilon^2$, t_k and y_k are turning points, T_η is the period of the oscillation along the coordinate η, and $\gamma = \overline{\eta^{-1}}/\overline{\xi^{-1}}$. The expectation values are to be understood in the sense in (3). We are using atomic units and the reduced variables

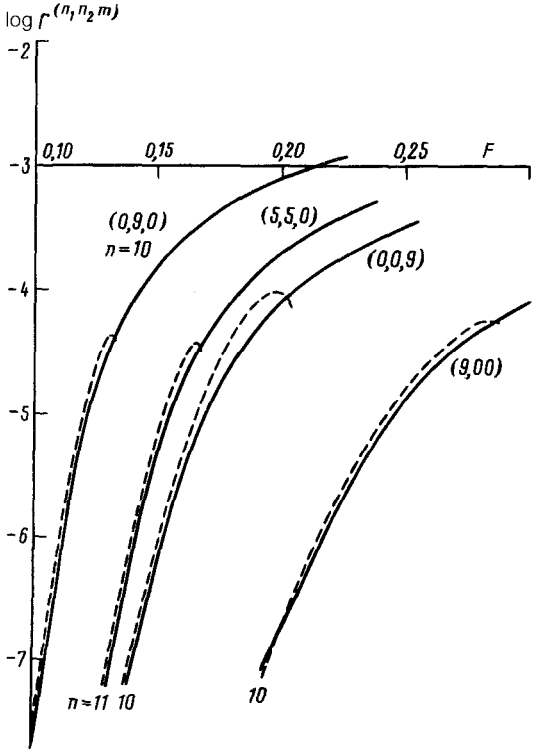

FIG. 1. Width of the levels in a hydrogen atom versus the reduced field F ($\hbar = e = m_e = 1$). The curves are labeled with the parabolic quantum numbers n_1, n_2, m and also with the value of n.

$$F = n^4\mathcal{E}, \quad \epsilon = 2n^2(E_r - i\Gamma/2), \quad \mu = m/n. \tag{7}$$

For $m = 0$ and also for states $(0, 0, n-1)$ with $n \gg 1$, all the quantities in (5) can be calculated analytically. The results of the calculations are shown in Fig. 1. The solid lines correspond to the Pade–Hermite approximant,[3)] and the dashed lines correspond to semiclassical formula (5), which is highly accurate if $a_\eta > 1$. The factor c, which is

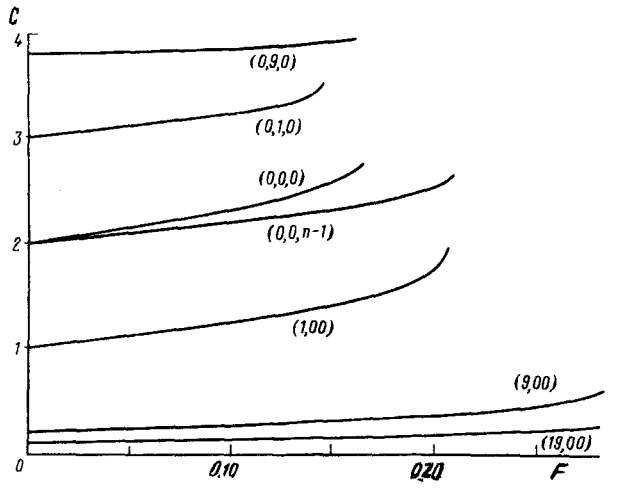

FIG. 2. The coefficient of the exponential function c [see Eq. (1)] for certain states (n_1,n_2,m).

associated with the multidimensional nature of the problem, is usually quite different from unity (Fig. 2), so the difference between (1), (5) and the one-dimensional Gamow formula is extremely large.

4. If $E \to U_m$, then $a \to 0$, and the oscillation period T diverges logarithmically. Approximation (1) then breaks down. It follows from Fig. 1 that the point at the maximum ($a = a_m$) of the dashed line is a natural boundary of the range of applicability of (1). We thus find the condition

$$a > a_m = [2\pi(\ln n_0 + b)]^{-1}. \tag{8}$$

Here n_0 is the number of states with energies $E < U_m$, and b is a constant which can be calculated (it depends on the particular problem). In the case of the Stark effect, for example, we would have $b = 2.16$, and we would replace n_0 by $n_2 + 1/2$. The parameter a_m is numerically small (by a virtue of the factor of $1/2\pi$) even at $n_0 \sim 1$. Consequently, the Gamow formula, like its generalization in (1), is valid except in a narrow energy interval near the top of the barrier.

[1)] Under the assumption $\exp(-2\pi a_i) \ll \exp(-2\pi a_f)$ with $i = \overline{1}, \ldots, f - \overline{1}$. In the semiclassical case, this condition is always satisfied (except, possibly, in systems having certain special symmetry properties).

[2)] The value of this constant depends on the particular problem [see, for example, Eq. (4)]. In contrast with α, the value of β_i is determined only along with the calculation of the energy E [in the one-dimensional case we would have $\alpha = c = 1$, and Eq. (1) would become the usual Gamow formula].

[3)] In other words, these results were calculated through a summation of divergent perturbation-theory series (in powers of $\mathscr{E}$) by the method of the Pade–Hermite approximant. See Refs. 5 and 6 for the details. At the accuracy level of this figure, the Pade–Hermite approximant coincides with the exact solution of the problem.

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Translated by D. Parsons