Nonperturbative quantum gravity as a theory of integrable hierarchies with constraints

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The theory of integrable hierarchies with constraints is examined. Applications of this theory to matrix models of nonperturbative quantum gravity and of matter interacting with it are discussed.

Nonperturbative quantum gravity, including a gravity interacting with matter, is described by matrix models,[1] which in turn generate[2] solutions of integrable hierarchies that obey certain auxiliary conditions.[3–6] The detailed relationships between specific matrix models and the various (reduced) integrable hierarchies, on the other hand, are not obvious. Different integrable hierarchies may be equally successful in nonperturbative descriptions of a certain class of theories with a 2D gravity, even if they do not have a genuinely matrix formulation. As an argument in support of this point of view, we show below how a correspondence with a field-theory description can be established through a purely "hierarchical" derivation of recursion relations[7] (or, in a slightly different context, loop relations[2,8]). Our method is based on a "dressing" procedure which is hidden in the structure of integrable systems.

We begin with a Toda hierarchy[9] with Virasoro constraints. A Toda hierarchy is universal among "discrete" (lattice) hierarchies, and in this case the relationship with matrix models, at least, is known.[5,10] Its "continuum" analog is a KP hierarchy. We suggest looking at Toda and KP hierarchies as instructive examples of general hierarchies with an r-matrix structure.[11] It seems very likely that since the loop equations derived below can be generalized to this broad class of integrable systems, there is a correspondence (equivalence?) of the type

(integrable hierarchies with constraints)↔(2D theories which are interacting with gravity),

to which we wish to call attention in this letter.

1. Very briefly, we recall the basic definitions pertinent to the Toda hierarchy.[9] The dressing operator W (or $\widehat{W}^{(\infty)}$, in the notation of Ref. 9) is a $Z\times Z$ matrix which acts in a vector space of the type $\Sigma_{s\in Z} v(s)|s\rangle$. We define the operators $\hat{p}$ and Λ by $\hat{p}|s\rangle = s|s\rangle$, $\Lambda|s\rangle = |s-1\rangle$. We then have

$$W = \sum_s |s><s|w(s;x,y;\Lambda),\quad W^{-1} = \sum_s w^*(s;x,y;\Lambda)|s><s|, \tag{1}$$

where $w(s;x,y;\lambda)$ and $w^*(s;x,y;\lambda)$ are expressed in terms of the τ-function by

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$$w(s;x,y;\lambda) \equiv \sum_{j=0}^{\infty} w_j(s;x,y)\lambda^{-j} = \frac{\tau(s;x-[\lambda^{-1}],y)}{\tau(s;x,y)},$$

$$w^*(s;x,y;\lambda) = \frac{\tau(s;x+[\lambda^{-1}],y)}{\tau(s;x,y)}. \tag{2}$$

Here $x=(x_1,x_2,x_3,...)$ specifies the times of the hierarchy, and $x \pm [\lambda^{-1}] = (x_1 \pm \lambda^{-1}, x_2 \pm \frac{1}{2}\lambda^{-2}, x_3 \pm \frac{1}{3}\lambda^{-3},...)$ (y designates yet another set of times, but a set in which we are not interested).

2. The initial "dynamic" data are the Virasoro constraints in the τ-function: $L_n\tau(x,y)=0$, $n\geqslant 0$, in which the Virasoro generators are given by the standard expressions

$$L_{p>0} = \frac{1}{2}\sum_{k=1}^{p-1}\frac{\partial}{\partial x_{p-k}}\frac{\partial}{\partial x_k} + \sum_{k\geq 1} kx_k\frac{\partial}{\partial x_{p+k}} + (\hat{p} + \frac{1}{2}(2J-1) + (J-\frac{1}{2})p)\frac{\partial}{\partial x_p},$$

$$L_0 = \sum_{k\geq 1} kx_k\frac{\partial}{\partial x_k} + \frac{1}{2}\left(\hat{p} + \frac{1}{2}(2J-1)\right)^2 - \frac{1}{24}, \tag{3}$$

and by similar expressions for $L_{p<0}$. Together, they form the algebra $[L_p,L_q] = (p-q)L_{p+q} + \delta_{p+q,0}(-p^3)(J^2-J+\frac{1}{6})$ (from which the role of the parameter J is clear). In terms of dressing operators, the Virasoro constraints can be rewritten in the equivalent form

$$\mathcal{L}_n \equiv \left(W\{[J(n+1)+\hat{p}]\Lambda^n + \sum_{r\geq 1} rx_r\Lambda^{r+n}\}W^{-1}\right)_- = 0, \quad n\geq 0, \tag{4}$$

where $(...)_-$ means the projection onto the lower-triangle part of the matrix. In this form, they are consistent with a scaling limit which is a "hierarchical" version[12] of a double-scaling limit.[1] In the limit $\epsilon\to 0$ we assume $s=t_1/\epsilon$, so that formally we have $\partial/\partial s=\epsilon D$, $D\equiv\partial/\partial t_1$ and $\Lambda = e^{\epsilon D}$. We introduce the following scaling ansatz:

$$w_j = \sum_{n\geq 1}\epsilon^n \begin{pmatrix} j+n-1 \\ n-1 \end{pmatrix} k_n, \quad j\geq 1, \quad w_i^* = \sum_{m\geq 1}\epsilon^m \begin{pmatrix} i+m-1 \\ m-1 \end{pmatrix} k_m^*, \quad i\geq 1. \tag{5}$$

Below, the k_i will become the coefficients of the dressing operator of the KP hierarchy. The temporal parameters and Virasoro generators are also subjected to scaling (cf. Ref. 10):

$$x_r = \frac{1}{r}\sum_{s=r}^{\infty}\begin{pmatrix} s \\ r \end{pmatrix}(-1)^{s+r}(s+1)\frac{t_{s-1}}{\epsilon^{s+1}}, \quad r\geq 1,$$

$$\tilde{\mathcal{L}}_{p-1} = \sum_{n=0}^{p}\begin{pmatrix} p \\ n \end{pmatrix}\mathcal{L}_n(-1)^{n+p}\epsilon^{-p+1}, \quad p\geq 0. \tag{6}$$

where t are new times (finite as $\epsilon \to 0$), and $\tilde{\mathcal{L}}$ are combinations of constraints which have a good scaling behavior. Assuming that k_l and k_m^* are finite functions of ϵ as $\epsilon \to 0$, and introducing the shift $t_1 \mapsto t_1 - \Sigma_{r\geqslant 2}(r+1)t_{t+1}(-1)^r \epsilon^{-r+1}$, we find new Virasoro constraints:[12]

$$\mathcal{L}_p^{(KP)} = \lim_{\epsilon \to 0} \tilde{\mathcal{L}}_p = (K(J(p+1) + \sum_{r\geq 1} rt_r D^r)D^p K^{-1})_- = 0, \quad p \geq -1, \tag{7}$$

where $K = 1 + \Sigma_{n \geqslant 1} k_n D^{-n}$ is the dressing operator of the KP hierarchy.[14] Now $(...)_-$ means a projection of pseudodifferential operators onto purely integral operators. [There is no simple *a priori* relationship between the two operations $(...)_-$ in the discrete and continuum cases.] The expressions which vanish according to (7) are precisely the Virasoro generators on the KP hierarchy, which were found directly in Ref. 13.

3. We have thus started with a discrete hierarchy with Virasoro constraints and taken its scaling limit. The continuum hierarchy turns out to obey its own Virasoro constraints. At this point, we see that the latter constraints provide an analogy (more precisely, a prototype for the case of an infinite number of prime fields) among the loop/recursion/puncture equations.[3,7,8] Specifically, constraints (7) on the KP hierarchy are summed in the generating expression

$$(K(P + lJ)e^{Dl}K^{-1})_- = 0, \quad P \equiv \sum_{r\geq 1} rt_r D^{r-1}, \tag{8}$$

where l plays the role of a loop parameter—a length. Incidentally, we note the equivalent formulation

$$K(x) \circ (x + lJ + \sum_{s\geq 2} st_s D^{s-1}) = A \circ K(x+l), \tag{9}$$

where $A = A(x,l)$ is a differential operator (of order N if the times t_r are nonzero only at $r \leqslant N+1$) which satisfies the system of nonlocal integrable equations

$$\frac{\partial A}{\partial t_r} = Q(x)_+^r A - AQ(x+l)_+^r. \tag{10}$$

It has been suggested[16] that evolution equations of this type be thought of as the result of a "quantization" of a spectral parameter in ordinary local hierarchies. It is interesting to compare that idea with the "quantum" Riemann surfaces from Ref. 4.

Going back to (8), we note that in addition to that equation, there are some "higher-order" loop relations which reflect the presence of a fairly large algebra for the symmetry of the KP hierarchy with Virasoro constraints. This algebra is the same as the "Borel subalgebra"[1] of the infinite W-algebra $W_\infty(J)$ (Ref. 8). [The J dependence is illustrated by the circumstance that a typical subalgebra—the higher-spin algebra[19] $B_\lambda = Usl(2)/I_\lambda$—is a factor of the universal wrapping $sl(2)$ with respect to the ideal generated by the relation (Casimir) $= \lambda \equiv J - J^2$.]

The complete set of W_∞ relations on the dressing operators is, for $J = 0$,

$$\exp\left\{\sum_{r\geq 2} t_r\left[\left[\frac{\partial}{\partial l}+\epsilon\right]^r-\frac{\partial^r}{\partial l^r}\right]\right\}(Ke^{\epsilon z}e^{lD}K^{-1})_- = (Ke^{lD}K^{-1})_- . \tag{11}$$

[The J dependence is reconstructed by formally incorporating the conjugation $e^{JlD}\ldots e^{-JlD}$ in the dressing operation, as becomes clear by rewriting of (8) in the form $(Ke^{JlD}Pe^{lD}e^{-JlD}K^{-1})_- = 0$.] Equation (11) is the Laplace transform of the constraints:

$$\sum_{n\geq 0} v^{-n-1}\oint dz z^n \mathfrak{D}(z+\mathcal{E}, z) = 0,$$

where the operator

$$\mathfrak{D}(u,v) = \psi(t,u)\circ D^{-1}\circ\psi^*(t,v) \tag{12}$$

[in which $\psi(t,u)$ and $\psi^*(t,u)$ are the wave function and the conjugate wave function of the KP hierarchy] has the clear geometric meaning of a "bc insert"[15] against the background[20] of the operator K. More precisely, the insertion is made by the vector field $\mathscr{D}(u,v)$ which is associated with $\widehat{\mathscr{D}}(u,v)$, on the space of operators K which perform an infinitesimal variation, $\delta K = \mathscr{D}(u,v)K$.

4. Theories with a finite number of "primary fields" follow from the discussion above through an N-reduction. We recall[14] that a reduction of a KP hierarchy by itself—without any constraints—to an N-KdV hierarchy can be achieved by imposing the requirement $(KD^NK^{-1})_- = 0$. The compatibility of this reduction with the symmetries of the KP hierarchy[21] which are generated by bilocal operators (12) is determined by the commutator of the corresponding symmetries and constraints of the vector fields on the space of operators K. These commutators reduce[15] to the commutators of the "bare" generators:

$$(K[e^{(u-v)P}\frac{1}{v}\delta(v,D), D^N]K^{-1})_- = (Ke^{(u-v)P}(v^N-u^N)\frac{1}{v}\delta(v,D)K^{-1})_-,$$

where $\delta(u,v) = \Sigma_{n\in z}(u^n/v^n)$ is a formal δ-function.[14] The vanishing condition thus gives us $u^N = v^N$, so we can set $u = z_a$, $v = z_b$, where $z_a = e_a z$ and $e_a = \exp[2\pi\sqrt{-1}(a/N)]$. We also recall that the spectral parameter of the N-KdV hierarchy lies on the complex curve $\{(z,E)\in C^2|z^N = P(E)\}$, where P is a polynomial. A projection onto $CP^1\ni E$ is an N-sheet tiling and makes it possible to introduce N wave functions $\psi^{(a)} = \psi[t,(z_a,E)]$ and, correspondingly, $\psi^{(a)*} = \psi^*[t,(z_a,E)]$. The symmetries of the N-KdV hierarchy which are induced from the KP hierarchy are thus generated by the "currents" (note the significant analogy with conformal field theories on Z_N curves[21])

$$\mathcal{J}^{ab}(E) = \psi^{(a)}(t,E)\circ D^{-1}\circ\psi^{(b)*}(t,E). \tag{13}$$

Vector fields associated with these pseudodifferential operators satisfy the current algebra $sl(N)$:

$$[[\hat{\mathcal{J}}^{ab}(E),\hat{\mathcal{J}}^{cd}(E')]] = \delta^{bc}\frac{1}{z}\delta(z,z')\hat{\mathcal{J}}^{ad}(E) - \delta^{ad}\frac{1}{z}\delta(z,z')\hat{\mathcal{J}}^{cb}(E), \tag{14}$$

where $[[,]]$ means the commutator of vector fields on operators, not of simply operators.

This current algebra $sl(N)$ may have an important interpretation in the context of W_N gravity. Since the currents in (13) satisfy (with respect to the brackets $[[,]]$) standard commutation relations with the "energy-momentum tensor"

$$\sum_{c=0}^{N-1} e_c^2 \frac{\partial \psi^{(c)}}{\partial z_c} \circ D^{-1} \circ \psi^{*(c)}, \tag{15}$$

which is induced from the KP hierarchy (more on this below), it is possible to impose constraints of higher weight simultaneously—for both a Virasoro algebra and a current algebra. In general, we are invoking the idea of (first) postulating suitable auxiliary constraints without deriving them from other principles and (only then) seeking a relationship with a field-theory description, utilizing for this purpose the "loop" equations derived in terms of hierarchies with auxiliary constraints. Experience over the past few decades indicates that the constraints of higher weight with respect to the semidirect product of Virasoro and Kac–Moody algebras will at the very least be substantive.

The complete symmetry algebra of the N-KdV hierarchy with Virasoro constraints can be found directly from the KP hierarchy with the constraints in (7): Only the generators $\mathscr{L}_n$ with $n = N_j, j \geqslant 0$ (and J should be set equal to zero), are compatible with the conditions for an N-reduction, so it is necessary to weaken the Virasoro constraints on the KP hierarchy to $\mathscr{L}_{Nj} = 0$. However, there is still more: After the N-reduction constraints are imposed, one can also impose the requirement $\mathscr{L}_{-N} = 0$ in a noncontradictory way. This requirement ultimately leads to an N-KdV hierarchy with Virasoro $\mathscr{L}_{(\geqslant 1)}$ constraints, where the genuinely N-KdV Virasoro generators are

$$\mathcal{L}_{(j)} = \frac{1}{N}(KPD^{Nj+1}K^{-1})_-, \quad P = \sum_{a,i}(Ni+a)t_{a,i}D^{Ni+a-1},$$

$$t_{a,i} = t_{Ni+a}, \quad i \geq 0, \quad a \in \{1,\ldots,N-1\}. \tag{16}$$

[These are the generators which make up energy-momentum tensor (15).] The complete system of constraints generated by the Virasoro constraints $\mathscr{L}_{(j)} = 0, j \geqslant -1$ can be written as a generating relation like (11):

$$(K(e^{\epsilon PD^{1-N}} - 1)e^{lD^N}K^{-1})_- = 0. \tag{17}$$

Since only the operator D^N is present in the "kernel" $\exp lD^N$, the first $N-1$ terms are singled out in the expression $PD^{1-N} = \Sigma_{a,i}(Ni+a)t_{a,i}D^{n(i-1)+a}$. As we know, these terms correspond to the primary fields.

[1] In other words, one which is generated by the generators of the higher spins $W_n^{(s)}$, for which $n \geqslant 8 - s + 1$.

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