Gaussian manifolds in random media

E. B. Kolomeisky
Institute of Crystallography Academy of Sciences of the USSR, 117333, Moscow

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Pis'ma Zh. Eksp. Teor. Fiz. **54**, No. 1, 52–57 (10 July 1991)

A unifying picture of scaling properties of D-dimensional Gaussian manifolds embedded in d-dimensional random media is presented. It is demonstrated, in particular, that for the special case of uncorrelated disorder manifold is stretched for $D > 2d/(4+d)$. The phase transitions between different stretched states and from the Gaussian form to the stretched form are predicted for $2d/(2+d) < D < 2$ and $0 < D < 2d/(4+d)$, respectively.

The properties of various manifolds (e.g., interfaces, membranes, polymers, etc.) interacting with random environment are the subjects of great importance in many branches of physics, chemistry, and biology. Despite their practical relevance and intrinsic interest, the theoretical understanding of these problems is still incomplete.

In the present letter we give a comprehensive description of the scaling properties of the above objects using a Flory-like approach invented by Zhang[1] in a different context.

We begin with the D-dimensional generalization of the Edwards model[2] for polymers, disregarding the self-avoidance effects. Consider a configuration $h(\rho)$ of a D-dimensional manifold immersed in a d-dimensional random medium. It can be described by the partion function

$$Z(h,\rho) = \exp(-F/T) = \int Dh \exp\{-\int d^D\rho(\frac{\Gamma}{2}|\nabla_D h(\rho)|^2 + V(h)/T)\} \quad (1)$$

where F is the free energy, T is the temperature, Γ is the manifold stiffness, $V(h)$ is the random potential with zero mean, whose properties are specified by its correlation function $\langle V(h)V(0)\rangle = \Delta^2 R_a(h)$, where Δ^2 is disorder strength $R_a(h)$ is a function of characteristic width a [for scales $h<a$ the random potential is strongly correlated and $R_a(h<a) = \text{const}$]. The finiteness of a plays an important role, as will be shown below. In what follows we assume that $R_a(h \geqslant a) = h^{\alpha}$. The problem formulated involves another intrinisic scale l_d which follows simply from dimensional arguments. For a manifold of linear size ρ and total size h the two terms in the Hamiltonian in (1) scale as $\Gamma\rho^{D-2}h^2$ and $\Delta\rho^D h^{\alpha/2}/T$, from which we can form a dimensionless coupling constant $g(h) = \Delta\rho^D h^{\alpha/2}/T\Gamma\rho^{D-2}h^2$. In the absence of defects the manifold is Gaussian, i.e., $\Gamma\rho^{D-2}h^2 \simeq 1$. Substituting this expression in the above determination of g, we obtain

$$g = (h/l_d)^{[4D+\alpha(2-D)]/2(2-D)}; \qquad l_d \simeq (T/\Delta\Gamma^{D/(2-D)})^{2(2-D)/[4D+\alpha(2-D)]}. \quad (2)$$

Note that $g(a)$ controls the dimensionless disorder strength.

For the special case $D=1$ the partion function (1) obeys the Schrödinger-like diffusion equation (where ρ is the time) and describes diffusion in a random environment (see the review articles in Refs. 2 and 3). Therefore, the comparison with some rigorous results will be possible.

We are interested in the scaling dependence of the moments $\langle Z^n\rangle$ and the manifold size $h \simeq A_\rho^{\nu}$. The free energy fluctuation ΔF is also of interest. It is customarily assumed that

$$\Delta F/T \simeq \Gamma A^2 \rho^{D-2+2\nu}, \quad (2)$$

which follows from a dimensionality argument about the elastic term $(\Gamma\rho^D(h/\rho)^2 \simeq \Gamma A^2\rho^{D-2+2\nu})$ in (1). Let us assume that the partion function (1) is the nth power and average it over the disorder

$$< Z^n > = < \exp(-nF/T) > = \int Dh \exp\{-(\Gamma/2)\int d^D\rho \sum_{i=1}^{n} |\nabla_D h_i|^2$$
$$+(\Delta^2/2T^2)\int\int d^D\rho d^D\rho' \sum_{i,j}^{n} R_a(|h_i - h_j|)\} . \quad (3)$$

It describes n identical manifolds, or replicas, with mutual interaction $R_a(h)$. In what follows we shall calculate $\langle Z^n\rangle$ with exponential accuracy, omitting all the numerical factors. However, before actually tackling the above path integral let us demonstrate, following Zhang[1] how to obtain the exponent ν and prefactor A. Suppose (3) is of the form $\exp(Bn^{\beta}\rho^{\gamma})$, i.e.,

$$\int dF P(F)\exp(-nF/T) = <\exp(-nF/T)> = \exp(Bn^{\beta}\rho^{\gamma}) \ , \tag{4}$$

then (4) can be interpreted as the Laplace transform of the probability distribution density $P(F)$ of the free energy of a single manifold in random media. Inverting (4) permits us to find the free energy distribution density

$$P(F) \cong \exp[-(\Delta F/T(\rho^{\gamma}B)^{1/\beta})^{\beta/(\beta-1)}],$$

where we set $\langle F\rangle = 0$. The knowledge of $P(F)$ enables us to determine the free energy fluctuation $\Delta F/T \simeq (B\rho^{\gamma})^{1/\beta}$. Comparing it with (2), we finally obtain

$$A^2 \simeq B^{1/\beta}/\Gamma; \qquad \nu = (2-D)/2 + \gamma/2\beta \ . \tag{5}$$

The last identity was first obtained by Zhang.[1] From (3) we have estimated for the free energy of n replicas:

$$F_n/T \simeq n(\Gamma\rho^{D-2}h^2 + \rho^D/\Gamma^{D/(2-D)}h^{2D/(2-D)}) - n^2\Delta^2\rho^{2D}h^{\alpha}/T^2, h \geq a. \tag{6}$$

The first term is the elastic energy of the distorted manifold, the second term (which is relevant for $0<D<2$ only) is an entropic repulsion among replicas confined into a well of characteristic size h.[2,4] The third term is defect-induced interaction among replicas. For $\Delta^2 = 0$ minimization of (6) with respect to h leads naturally to the Gaussian manifold

$$h_G^2(\rho) \simeq \rho^{2-D}/\Gamma. \tag{7}$$

For $\alpha > 0$ the third term in (6) corresponds to the repulsion between replicas. In this case the minimum in (6) for $\rho \to \infty$ is determined by the first and third terms: $h \cong (n\Delta^2\rho^{D+2}/T^2\Gamma)^{1/(2-\alpha)}$. This solution is valid in the region $0 \leqslant \alpha < 2$. Substituting it into (6), we obtain the following expression:

$$\ln <Z^n> \simeq (\Delta^4/T^4\Gamma^{\alpha})^{1/(2-\alpha)} n^{(4-\alpha)/(2-\alpha)} \rho^{(4D-D\alpha+2\alpha)/(2-\alpha)}, \tag{8}$$

$$h \cong (\Delta/\Gamma T)^{2/(4-\alpha)} \rho^{4/(4-\alpha)}. \tag{9}$$

Expression (9), however, is not asymptotic. For positive α $\nu > 1$; i.e., the Hamiltonian in (1) is unstable with respect to adding infinitely many gradient terms. However, this change in the partion function (1) appears only at $h \simeq \rho$.

For negative α it is useful to change $\alpha = -d$ in order to include the case of δ-correlated disorder. If $2D/(2-D) > d$ and $0<D<2$, or $2d/(2+d) < D < 2$, Eq. (6) has a single minimum $h \simeq \max\{(T^2/\Delta^2 n\rho^D\Gamma^{D/(2-D)})^{(2-D)/[2D-d(2-D)]}, a\}$. If

$$(T^2/\Delta^2 n\rho^D\Gamma^{D/(2-D)})^{(2-D)/[2D-d(2-D)]} > a \ , \tag{10}$$

straightforward algebra gives the results

$$\ln < Z^n > \simeq \{(\Delta/T)^{4D}\Gamma^{Dd}(n\rho^D)^{4D-d(2-D)}\}^{1/[2D-d(2-D)]}, \quad (11)$$

$$h \simeq \{(\Delta/T)^{2D}\Gamma^{d-2D}\}^{1/[4D-d(2-D)]}\rho . \quad (12)$$

Note that (11) coincides with the available exact solution[5] for the special case $a=0$, $d=D=n=1$. If (10) is broken, we obtain

$$\ln < Z^n > \simeq \Delta^2 n^2 \rho^{2D}/T^2 a^d, \quad (13)$$

$$h \simeq (\Delta/T\Gamma a^{d/2})^{1/2}\rho. \quad (14)$$

For $D=1$, Eq. (13) was initially derived by Zeldovich *et al.*[6]

If $0<D<2d/(2+d)$, Eq. (6) has two minima $h_1 = h_G(\rho)$ (7) and $h_2 \simeq a$. Substituting (7) into (6), we obtain

$$F_n/T \cong n - \Delta^2\Gamma^{d/2}T^{-2}n^2\rho^{2D-d(2-D)/2}.$$

It is clear that the results depend on the sign of $4D-d(2-D)$. For $2d/(4+d)<D<2d/(2+d)$ we find

$$\ln < Z^n > \simeq \Delta^2\Gamma^{d/2}T^{-2}n^2\rho^{2D-d(2-D)/2}, \quad (15)$$

$$h \simeq (\Delta\Gamma^{d/4-1}/T)^{1/2}\rho^{1-d(2-D)/8}. \quad (16)$$

Comparing (15) and (13), we conclude that the solution is stable only for scales $\rho < (\Gamma a^2)^{1/(2-D)}$. In the case of opposite inequality, we return to (13) and (14).

The solution (13), (14) is also stable for $0<D<2d(4+d)$ if inequality (10) is valid. If (10) is broken, the Gaussian manifold solution (7) is stable. For $D>2$ expression (6) contains the only minimum $h \simeq a$ and we return results (13), (14). Let us discuss now how the replica results manifest themselves in the original random system, i.e., in the limit $n \to 0$. The situation for $-d=\alpha>0$, $D>2$ and $2d/(4+d)<D<2d/(2+d)$ is clear: the manifold is stretched according to (9) and (14) [with the crossover regime (16), (17) for $2d/(4+d)<D<2d/(2+d)$], respectively. Other cases $0<D<2d/(4+d)$ and $2d/(2+d)<D<2$ are connected with the inequality (10) which cannot be continued to $n=0$ in a straightforward way. However, this problem has a simple solution. Indeed, the probability distribution function $P(F)$ in (4) is obtained for large ρ. Therefore, (4) in fact imposes a relationship between the saddle-point value of n and ρ: for $n \to 0$, $\rho \to \infty$, we need $Bn^\beta\rho^\gamma \simeq 1$. The results which follow from this rule and from (7) and (10)–(14) can be conveniently represented in terms of $g(a)$ (2)(for $D=1$ similar results were obtained by Nattermann and Renz[3]).

For $2d(2+d)<D<2$ one predicts the phase transition between different stretched states: from (11), (12) [for $g(a)<1$] to (13), (14) [for $g(a)>1$].

For $0<D<2d/(4+d)$ the phase transition from the Gaussian manifold (7) [for $g(a)<1$] to the stretched manifold (13), (14) [for $g(a)>1$] takes place.

In conclusion, let us discuss the range of validity of the results obtained. First, the path integral (3) was estimated using the saddle-point approximation. Such approach differs from the rigorous solution only in the sub-leading-order terms for large ρ. Moreover, this estimate was satisfied with the help of a Flory-like arguments.[1–3] However, such an approximation is essentially uncontrolled.[2] Therefore, an independent treatment is necessary to confirm (or reject) the Flory-type results. We believe that our method gives the correct (if not exact) values of the exponents, since it reproduces the exact result[5] for $a = 0$, $d = D = n = 1$ and the results obtained independently for $D = 1$.

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