Self-dual solutions of the Yang–Mills equations for an arbitrary gauge group

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The self-duality equations for gauge fields of an arbitrary semisimple Lie group G are reduced to a system of equations which breaks up into the equations $\Box\varphi = 0$ for the scalar field φ and Nahm equations (nonlinear ordinary differential equations).

1. In this letter we would like to introduce a generalization of the Corrigan–Fairlie–'t Hooft–Wilczek ansatz[1] and describe some new classes of solutions which correspond to this ansatz.

In Euclidean space $R^{4,0}$ we consider a gauge field A_μ of an arbitrary semisimple Lie group G with a stress $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, $\mu,\nu,... = 1,...,4$. For self-dual gauge fields, the components $F_{\mu\nu}$ are by definition sent into themselves by the dual-conjugation operator *:

$$*F_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma} = f_{\mu\nu}. \tag{1}$$

The A_μ automatically satisfy Yang–Mills equations by virtue of the Bianchi identities.

More convenient here is the following form of the self-duality equations:[2]

$$\bar\eta^a_{\mu\nu}F_{\mu\nu} = 0. \tag{2}$$

Here the $\bar\eta^a_{\mu\nu}$ are components of the anti-self-dual $(*\bar\eta^a_{\mu\nu} = -\bar\eta^a_{\mu\nu})$ 't Hooft tensor, which are given by $\bar\eta^a_{bc} = \epsilon^a_{bc}$, $\bar\eta^a_{\mu\nu} = -\delta^a_\mu$, and $\bar\eta^a_{\mu\nu} = -\bar\eta^a_{\nu\mu}$, where ϵ^a_{bc} are structure constants for the Lie group $SU(2)$, and $a,b,... = 1,2,3$.

2. We consider the following ansatz for A_μ:

$$A_\mu = \bar\eta^a_{\mu\lambda}T_a(\varphi)\partial_\lambda\varphi, \tag{3}$$

where the φ-dependent functions $T_a(\varphi)$ take on values in the Lie algebra $\mathscr{G}$ of the semisimple Lie group G, and φ is an arbitrary function of the coordinates x_μ. Ansatz (3) generalizes the Corrigan–Fairlie–'t Hooft–Wilczek ansatz,[1] becoming the latter when we adopt the very simple functional dependence $T_a(\varphi) = -(1/\varphi)J_a$ with constant matrices J_a which satisfy the commutation relations of the $SU(2)$ Lie group: $[J_a, J_b] = \epsilon_{abc}J_c$.

Substituting (3) into the definition of $F_{\mu\nu}$, and using some identities for $\bar\eta^a_{\mu\nu}$ (Ref. 2),

$$\epsilon^a_{bc}\bar\eta^b_{\mu\lambda}\bar\eta^c_{\nu\sigma} = \delta_{\mu\nu}\bar\eta^a_{\lambda\sigma} - \delta_{\mu\sigma}\bar\eta^a_{\lambda\nu} - \delta_{\lambda\nu}\bar\eta^a_{\mu\sigma} + \delta_{\lambda\sigma}\bar\eta^a_{\mu\nu},$$

$$\bar\eta^a_{\mu\nu}\bar\eta^b_{\mu\lambda} = \delta^{ab}\delta_{\nu\lambda} + \epsilon^{abc}\bar\eta^c_{\nu\lambda}. \tag{4}$$

we find

$$F_{\mu\nu} = \bar{\eta}^a_{\nu\lambda}[T_a \partial_\lambda \partial_\mu \varphi + (\dot{T}_a + \epsilon_{abc} T_b T_c) \partial_\lambda \varphi \partial_\mu \varphi]$$

$$\bar{\eta}^a_{\mu\lambda}[T_a \partial_\lambda \partial_\nu \varphi + \dot{T}_a + \epsilon_{abc} T_b T_c) \partial_\lambda \varphi \partial_\nu \varphi]$$

$$\bar{\eta}^a_{\mu\nu} \epsilon_{abc} T_b T_c \partial_\lambda \varphi \partial_\lambda \varphi, \tag{5}$$

where $\dot{T}_a \equiv dT_a/d\varphi$.

We assume $\varphi = x_\mu x_\mu$, and we assume that the matrices $T_a(\varphi)$ satisfy Nahm equations:[3]

$$\dot{T}_a = -\epsilon_{abc} T_b T_c. \tag{6}$$

It is easy to see that in this case the tensor $F_{\mu\nu}$ becomes anti-self-dual:

$$F_{\mu\nu} = -4\bar{\eta}^a_{\mu\nu}(T_a - x_\lambda x_\lambda \epsilon_{abc} T_b T_c). \tag{7}$$

Consequently, corresponding to each solution of Nahm equations (6) with $\varphi = x_\mu x_\mu$ is an anti-self-dual solution (7) of the Yang–Mills equations for an arbitrary semisimple Lie group G. This class of solutions was found by the present author in collaboration with T. A. Ivanova.

We now substitute (5) into self-duality equation (2). Using the Jacobi identity for the matrices T_a, along with identities (4), we find that (2) reduces to the equations

$$T_a \Box \varphi + (\dot{T}_a - \epsilon_{abc} T_b T_c) \partial_\lambda \varphi \partial_\lambda \varphi = 0. \tag{8}$$

The tensor $F_{\mu\nu}$ is evidently self-dual if the following equations hold simultaneously:

$$\dot{T}_a = \epsilon_{abc} T_b T_c, \tag{9a}$$

$$\Box \varphi = 0. \tag{9b}$$

The $\pm$ signs in Nahm equations (6) and (9a) are unimportant, since they can easily be changed through the substitution $T_a \to -T_a$.

3. Any solution of Eqs. (9) gives us a self-dual solution of the type in (3) of the Yang–Mills equations. All solutions of linear equation (9b) are known; the only thing left to do is to point out the solutions of Nahm equations (9a).

The Nahm equations were studied in Refs. 3–5. Their solutions are used in constructing multimonopole solutions for the $SU(2)$ group by the Nahm algorithm.[3–5] It is a rather difficult matter to construct explicit general solutions of Eqs. (9a), so we will simply point out classes of particular solutions.

The simplest solution of the Nahm equations is found by setting $T_a = -(1/\varphi)J_a$, $[J_a, J_b] = \epsilon_{abc} J_c$. The effect is to reduce ansatz (3) to the Corrigan–Fairlie–'t Hooft–Wilczek ansatz and to reduce the solutions to the 't Hooft solutions.

As a particular case we find N-instanton solutions for the $SU(2)$ group. In addition, any solution $T_a(\varphi)$ of the Nahm equations incorporates the solution $T_a(\varphi) = -(1/\varphi)J_a$ as a continuous limit (in terms of the parameters).

Another class of solutions can be found by noting that the Nahm equations can be reduced to the equations of the Toda chain. Equations of a generalized Toda chain were introduced for an arbitrary Lie algebra by Bogoyavlensky.[6] Using those results (see also the review in Ref. 7), we can easily write an ansatz for T_a in terms of the Cartan–Weyl basis of the Lie algebra, $\mathscr{G}^{\vec{\mathbb{C}}} = \mathscr{G} \otimes \vec{\mathbb{C}}$. As an example, we write an explicit expression for the ansatz for matrices T_a with values in the Lie algebra $\mathscr{G} = su(n)$. Following Ward,[9] we set

$$T_1 = i\begin{pmatrix} 0 & a_1 & \cdots & 0 & a_n \\ a_1 & 0 & & & 0 \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & 0 & \ddots & \vdots \\ 0 & & & 0 & a_{n-1} \\ a_n & 0 & \cdots & a_{n-1} & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & a_1 & \cdots & 0 & -a_n \\ a_1 & 0 & & & 0 \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & 0 & \ddots & \vdots \\ 0 & & & 0 & a_{n-1} \\ a_n & 0 & \cdots & a_{n-1} & 0 \end{pmatrix},$$

$$T_3 = i\begin{pmatrix} b_1 & & & & & \\ & b_2 & & & 0 & \\ & & \ddots & & & \\ & & & \ddots & & \\ & 0 & & & b_{n-1} & \\ & & & & & b_n \end{pmatrix}, \tag{10}$$

where $\Sigma_{i=1}^n b_i = 0$, and $a_i = a_i(\varphi)$ and $b_i = b_i(\varphi)$ are real functions which depend on φ. It is easy to see that the equations $\dot{T}_1 = [T_2, T_3]$ and $\dot{T}_2 = [T_3, T_1]$ are the same for ansatz (10). We introduce the matrices $L = T_2 - iT_3$ and $M = iT_1$, which take on values in the Lie algebra $sl(n,R)$. The Nahm equations can then be written as a Lax equation which describes an ordinary periodic Toda chain:

$$\dot{L} = [L, M]. \tag{11}$$

It we set $a_n = 0$ in (10), Eqs. (11) become the equations of an ordinary (finite) aperiodic Toda chain.

All the solutions of the equations of a Toda chain, (11), are known (Refs. 7 and 8, for example), but we do not have room to write them out here. The solutions of the equations of the Toda chain and Eqs. (9b) thus provide a large class of self-dual solutions of the Yang–Mills equations. In the Nahm algorithm,[3–5] an n-monopole solution of the Yang–Mills equations with the $SU(2)$ group is then associated with each solution of Eqs. (9a) for $T_a \in su(n)$ satisfying certain restrictions. It would be interesting to learn whether the solutions of the equations of a periodic Toda chain satisfy these restrictions. In addition, it would be interesting to find non-self-dual solutions of the Yang–Mills equations, of the meron type, for ansatz (3).

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