Breaking of a Riemann wave in dispersive hydrodynamics

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A general method is developed for analytically solving Whitham's modulation equations, which describe the structure of a dissipationless shock wave after an arbitrary monotonic profile breaks in a Korteweg–de Vries hydrodynamics.

1. A simple Riemann wave is described by the equation $\partial_t r + V(r)\partial_x r = 0$, which has the solution

$$x - V(r)t = W(r), \tag{1}$$

where the function $W(r)$ is the inverse of the initial profile $r = r_0(x)$. After a simple wave breaks in a Korteweg–de Vries hydrodynamics, the solution is described by the three functions r_1, r_2, r_3 (Fig. 1), which satisfy Whitham's modulation equations

$$\partial_t r_i + V_i(r)\partial_x r_i = 0. \tag{2}$$

The characteristic velocities $V_i(r)$ are given by explicit expressions, and $r = (r_1, r_2, r_3)$ (Refs. 1–3).

A vector generalization of Riemann wave (1) for system (2) is

$$x - V_i(r)t = W_i(r), \tag{3}$$

but the function $W_i(r)$ is not arbitrary. It must satisfy compatibility conditions found by substituting (3) into (2):

$$\frac{\partial_i W_j}{W_i - W_j} = \frac{\partial_i V_j}{V_i - V_j} \quad i \neq j, \ \partial_i \equiv \partial_{r_i} \tag{4}$$

[the construction in (3), (4) is known as the generalized hodograph method[3,4]]. We would also obtain Eqs. (4) if we analyzed the system of equations

$$\partial_\tau r_i + W_i(r)\partial_x r_i = 0, \tag{5}$$

which specifies the flux $r_i(\tau)$ which commutes with the solution of (2), i.e., the flux $r_i(t)$. In other words, the general solution of compatibility equations (4) in r space describes all the r fluxes which commute with $r(t)$ from (2) (Ref. 3).

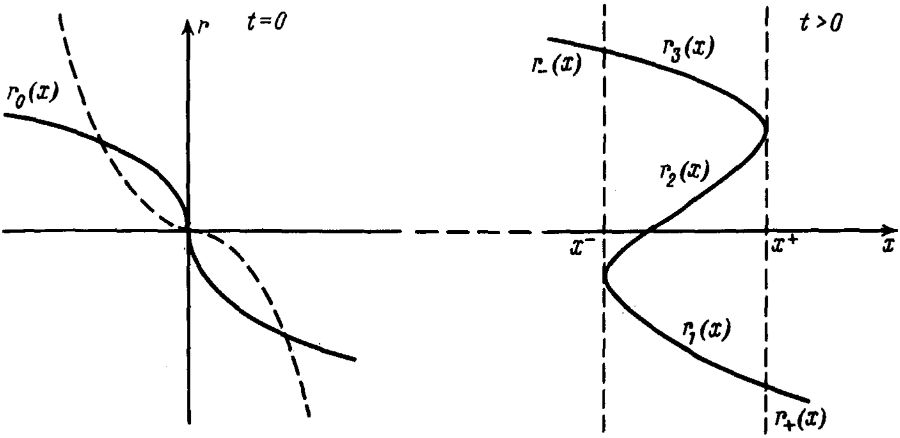

FIG. 1. Breaking of a simple wave $r(x, t)$ and formation of a dissipationless shock wave; behavior of the Riemann invariants as a function of x. The dashed line is the inverse of the initial profile $r = r_0(x)$.

2. As a result of Eqs. (2), the number of waves is conserved:[1]

$$\partial_t k + \partial_x (kU) = 0, \tag{6}$$

where k is the wave number, and U is the phase velocity. In the Korteweg–de Vries case we would have

$$U = \frac{1}{3}\sum_{j=1}^{3} r_j, \quad \lambda = \frac{2\pi}{k} = 6^{1/2}\int_{r_1}^{r_2} d\mu [\Pi_{j=1}^{3}(\mu - r_j)]^{-1/2} = \frac{6^{1/2}K(m)}{(r_3 - r_1)^{1/2}}. \tag{7}$$

Here $K(m)$ is the complete elliptic integral of the first kind, and $m = (r_2 - r_1)/(r_3 - r_1)$. Going over to the Riemann variables r_j in (6), we obtain a "potential" representation for the characteristic velocities:

$$V_i(r) = U + k\partial_i U/\partial_i k = U - \lambda \partial_i U/\partial_i \lambda. \tag{8}$$

Let us consider the following equation, which commutes with (6):

$$\partial_\tau k + \partial_x (kf) = 0, \tag{9}$$

where f is a generalized phase velocity [certain equations of the type in (9) have the natural meaning of conservation laws for the number of waves for higher-order Korteweg–de Vries situations]. By analogy with (8) we find the following expression for W_i in terms of the (scalar) function f:

$$W_i = f - \lambda \partial_i f/\partial_i \lambda = f + (V_i - U)\partial_i f/\partial_i U. \tag{10}$$

Substituting (8) and (10) into Eqs. (4), we find a scalar formulation of the compatibility equations:

$$\frac{\partial^2_{ij} f}{\partial_i f - \partial_j f} = \frac{\partial^2_{ij}\lambda}{\partial_i \lambda - \partial_j \lambda}, \quad i \neq j. \tag{11}$$

Using (7), we finally find

$$\frac{\partial^2_{ij} f}{\partial_i f - \partial_j f} = \frac{1}{2(r_i - r_j)}$$

or

$$E_{ij} f = 0, \text{ where } E_{ij} = \partial^2_{ij} - (\partial_i - \partial_j)/2(r_i - r_j). \tag{12}$$

Equations (12) were derived by a different method in Ref. 5. Each of these equations, for a given pair, i, j, is an Euler–Poisson equation in r_i, r_j at a fixed $r_k = r_{k0}$ $(k \neq i,j)$. Of importance to the discussion below are the homogeneous solutions of (12) of the form $f = r_i^q \Phi\ (-q, 1/2; 1/2 - q; r_j/r_i)$, where q is an arbitrary number (not necessarily an integer), and $\Phi(a, b; c; z)$ is the solution of the corresponding hypergeometric equation.[6]

3. Let us examine the problem of the breaking of simple wave (1) with a monotonic initial profile in a dispersive Korteweg–de Vries hydrodynamics. We assume that the breaking begins at $t = 0$ at the point $x = 0$, $r = 0$ (Fig. 1), where

$$r(x,\ 0) = \begin{cases} r_0^+(x) < 0 & \text{for} \quad x \geq 0, \\ r_0^-(x) > 0 & \text{for} \quad x < 0; \end{cases} \qquad W(r) = \begin{cases} W_+(r) & \text{for} \quad r \leq 0, \\ W_-(r) & \text{for} \quad r > 0. \end{cases} \tag{13}$$

The solution which we need describes the evolution of the dissipationless shock wave, which lies between the boundaries $x = x^-(t)$ (the trailing edge) and $x = x^+(t)$ (the leading edge). This solution satisfies certain conditions on the curves of $x^{\pm}(t)$. These conditions are that the "external" solution in (1), $r(x,t)$, join with the solution (r_1, r_2, r_3) of the "internal" modulation equations, (2) (Ref. 2):

$$r_3(x^-, t) = r_-(x^-, t) \quad \text{for} \quad r_2 = r_1; \quad r_1(x^+, t) = r_+(x^+, t) \quad \text{for} \quad r_2 = r_3. \tag{14}$$

Interestingly, when we go over to r space, conditions (14) take the simple form

$$W_1 = W_+(r_1) \quad \text{for} \quad r_2 = r_3; \quad W_3 = W_-(r_3) \quad \text{for} \quad r_2 = r_1. \tag{15}$$

As a result, instead of a problem with conditions at an unknown boundary, (14), in r space, linear system (4) satisfies simple linear conditions at given boundaries.[7] Switching to the scalar function f, and using (10) with $r_2 = 0$, we find the boundary conditions

$$\begin{aligned} f = f_-(r_3) &= \frac{1}{2} r_3^{-1/2} \int_0^{r_3} x^{-1/2} W_-(x) dx \quad \text{for} \quad r_1 = r_2 = 0, \\ f = f_+(r_1) &= \frac{1}{2}(-r_1)^{-1/2} \int_0^{-r_1} x^{-1/2} W_+(-x) dx \quad \text{for} \quad r_3 = r_2 = 0. \end{aligned} \tag{16}$$

The satisfaction of conditions (16) on the $r_2 = 0$ plane implies the satisfaction of (15) (for the regular solution). Combining Eqs. (4), we easily find the result

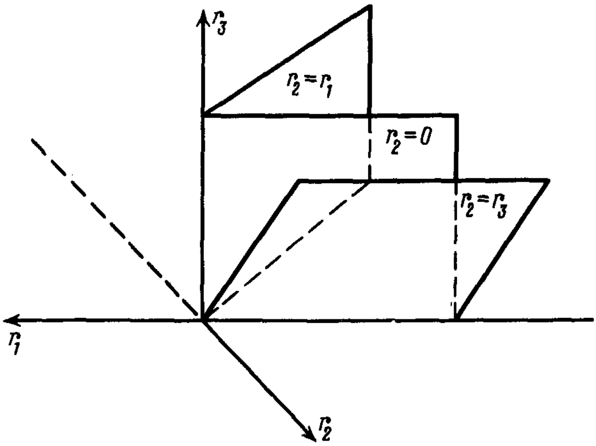

FIG. 2. Region in r space in which the solution is defined. Here $r_2=r_1$ is the trailing edge; $r_2=r_3$ is the leading edge; and $r_1<r_2<r_3$, $r_3>0$, $r_1<0$.

$\partial_1 W_1=(\partial_2+\partial_3)W_1/_{r_2=r_3}$ for arbitrary r_1. In other words, the function W_1 is constant along the leading edge (and corresponding comments apply to W_3 and the trailing edge).

For $f(r_1, r_2, r_3)$ we thus have system of equations (12) in the region $r_1\leqslant r_2\leqslant r_3$, $r_1\leqslant 0$, $r_3\geqslant 0$ between the $r_2=r_1$ and $r_2=r_3$ planes (Fig. 2), with boundary conditions (16). Let us construct a solution for this problem.

We first find $f(r_1, 0, r_3)$ from Goursat problem (12), (16):

$$E_{31}f=0, \quad f|_{r_1=0}=f_-(r_3), \quad f|_{r_3=0}=f_+(r_1). \tag{17}$$

Since problem (17) is linear, it is sufficient to solve it for the case (for example) $f_+(r_1)=0$. From the representation of $f_-(r_3)$ as a Mellin integral,[7]

$$f_-(r_3)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r_3^q S(q)dq, \quad S(q)=\int_0^\infty r_3^{-q-1}f_-(r_3)dr_3\,,$$

it is clear that we can restrict the analysis to $f_-^{(q)}(r_3)=r_3^q$. A solution of this problem can be written out explicitly:

$$f^{(q)}(r_1,r_3)=r_3^q\left[\frac{\Gamma(q+1)}{\Gamma(q+1/2)}\right]^{1/2}\frac{1}{q+1/2}u_4(r_1/r_3), \tag{18}$$

where $u_4(z)=z^{-1/2}F(q+1, 1/2; q+3/2; z^{-1})$ is the corresponding Kummer solution[6] of the hypergeometric equation with the parameters $(-q, 1/2; 1/2-q)$, and $F(a, b; c; z)$ is the Gauss hypergeometric function.

The solution found, $f(r_1, 0, r_3)$, can then be thought of as a condition (of the Goursat type) for corresponding boundary-value problems in the $r_1=\text{const}$, $r_2>0$

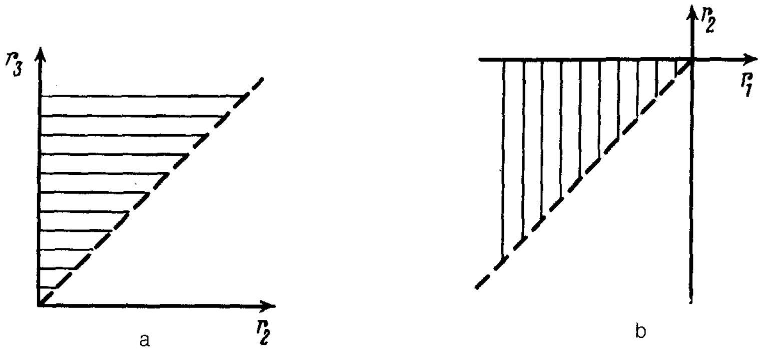

FIG. 3. Integration regions. a—In a plane with $r_1 = \text{const}$; b—in a plane with $r_3 = \text{const}$.

and $r_3 = \text{const}$, $r_2 > 0$ planes (Fig. 3). In planes with $r_1 = \text{const}$, for example, we have the following problem (r_1 appears as a parameter; $r_2 > 0$): We have $E_{32}f(r_1,r_2,r_3) = 0$, $f(r_1,0,r_3) = f^{(q)}(r_1,r_3)$ [see (18)], and $f(r_1,r_2,r_3)$ is regular at $r_2 = r_3$. By virtue of the linearity of the problem, it is again convenient to write $f^{(q)}(r_1,r_3)$ as a Mellin integral (or as a power series in r_3). The solution for each term has form like that of (18), with $u_4(z)$ replaced by[6] $u_6(z)$, which is regular at the leading edge ($z = 1$). A solution is found in the planes $r_3 = \text{const}$, $r_2 < 0$ in a corresponding way. The solution is thus constructed on the two sides of the $r_2 = 0$ plane, at which the function f is given. The continuity of the normal derivative $\partial_2 f$ at $r_2 = 0$ can be checked directly (cf. Ref. 2). The $x^{\pm}(t)$ curves, which bound the region of the dissipationless shock wave, are found through a joint analysis of solution (3) and the conditions $dx^{\pm}/dt = V^{\pm}$ at the boundaries ($V^{\pm}$ are multiple characteristic velocities).[2]

4. Examples. It is clear from the discussion above that, without any loss of generality, it is sufficient to analyze the breaking problem with initial data (13), where

$$r_0^-(x) = (-x)^{1/q_-},\quad r_0^+(x) = -x^{1/q_+};\quad W_-(r) = -r^{q_-},\quad W_+(r) = (-r)^{q_+},\quad q_\pm > 1. \tag{19}$$

With $q_+ = q_- = q$, the solutions $r_i(x, t)$ which we need are self-similar:[2,3] $r_i = t^{\gamma} l_i(x/t^{\gamma+1})$, where $\gamma = 1/(q-1)$.

a) A quasisimple wave $r = (0, r_2, r_3)$. We ultimately find a family of solutions with $r_1 = 0$, which were discussed in Ref. 7. A wave of this sort is described by the equation $E_{32}f(r_2,r_3) = 0$ with given $f(0,r_3) = -r_3^q/(2q+1)$, corresponding to a breaking of a monotonic profile $r_0^-(x) = (-x)^{1/2}$, $r_0^+(x) = 0$. The solution which we need, and which is regular on the bisector $r_2 = r_3$, is $f(r) = -r_3^q \pi^{1/2}\Gamma(1+q)u_6(r_2/r_3)/2\Gamma(q+3/2)$. For integer values $q = M$, the hypergeometric series is truncated, and the solution takes the symmetric form

$$f(r) = P_M(r_2, r_3) = -\frac{2^M M!}{(2M-1)!!(2M+1)} \sum_{k_2+k_3=M} \frac{(\frac{1}{2})_{k_2}(\frac{1}{2})_{k_3}}{k_2!k_3!} r_2^{k_2} r_3^{k_3},$$

$$(a)_n = \Gamma(a+n)/\Gamma(a).$$

b) Breaking of an antisymmetric profile $r = (r_1, r_2, r_3)$. We assume $q = M$ (an integer). Then for odd values of M we have

$$f(r) = P_M(r) = -\frac{2^M M!}{(2M-1)!!(2M+1)} \sum_{k_1+k_2+k_3=M} \frac{(\frac{1}{2})_{k_1}(\frac{1}{2})_{k_2}(\frac{1}{2})_{k_3}}{k_1!k_2!k_3!} r_1^{k_1} r_2^{k_2} r_3^{k_3}, \tag{20}$$

and for even values of M we have

$$f(r) = \begin{cases} -P_M(r) + I(-r_1, \ -r_2, \ r_3) & \text{for} \quad r_2 < 0 \\ P_M(r) - I(r_3, \ r_2, \ -r_1) & \text{for} \quad r_2 > 0, \end{cases}$$

Although an explicit integral representation of $I(r)$ is known, it is lengthy, and we will not reproduce it here. For even integer values of q, the solution is thus not a polynomial solution. As we mentioned earlier, the derivative $\partial_2 f(r_2 = 0)$ is continuous. Nevertheless, the $r_2 = 0$ plane is obviously singular. This result is not surprising: An initial profile with even q is not an analytic function, and it cannot be found as a result of the evolution of a smooth profile of a Riemann wave. Only a profile with odd values of q, for which solution (20) is of a polynomial nature and has no singularities, satisfies the evolution properties.

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