From instantons to inflationary universe

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The present paper is based on a theory which includes gravity and a complex scalar field.[1] In such a theory it is possible to analyze the evolution from instantons in the classically forbidden (Euclidean) region in minisuperspace to the inflationary universe in the classically allowed (Minkowski) region. The characteristics for the Hamilton–Jacobi equation, which define the action in the quasiclassical approximation, are described by four first-order differential equations. This four-dimensional dynamical system was integrated numerically. In the closed Euclidean region two types of instantons were found. It is shown that the instantons correspond to extremal trajectories. The existence of two types of instantons gives rise to different possibilities for tunneling from Euclidean region to Minkowski region and for creation of inflationary universes.

The well-known contemporary theory of an inflationary universe is based on a Friedmann homogeneous model in the presence of a real scalar field. In such a model it is possible to analyze all possible inflationary solutions for three types of geometry (flat, open, and closed), depending on the initial conditions which we fix on some surface in phase-space and which we define as the end of the quantum era.[2] At this initial surface the density of energy is on the order of the Planck energy, $\epsilon_p \sim m_p^4$.

The quantum creation of a universe in such a model was studied in Ref. 3. The only result which was possible to obtain was the creation of a large but empty universe (free from a field) which, as we know,[2] cannot inflate. In Ref. 1, we formulated a model which considers gravity (Friedmann closed model) induced by a complex scalar field. One reason to consider a complex, instead of real, scalar field is the fact that the energy-momentum tensor for such a field simply corresponds to the hydrodynamic energy-momentum tensor usually used in general relativity. The main result obtained in Ref. 1 is the existence of a closed area in minisuperspace (metric and field) which is classically forbidden and which has a partly convex and partly concave boundary. Because of this circumstance, there are two possible types of instantons which are able, after tunneling to the classically allowed region, to create the inflationary universe. The model of a complex scalar field coupled to gravity was previously used by Lee[2] to study wormhole physics, a problem different from ours.[1)]

We recall the basic equations of our model.[1] The action of gravitation and of a complex massive scalar field has the form[2)]

$$S=\int d^4x\sqrt{-g}\left(-\frac{M_p^2}{16\pi}R+\frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi^*_{,\nu}-\frac{1}{2}m^2\varphi\varphi'-\lambda\right), \tag{1}$$

where the metric $g_{\mu\nu}$ for a homogeneous closed model and the field φ we choose as follows:

$$\varphi=\sqrt{\frac{3}{4\pi}}M_p x(t)e^{i\Theta(t)}, \tag{2}$$

$$g_{\mu\nu}=\mathrm{diag}\left[\frac{N^2(t)}{m^2};-\left(\frac{2}{3\pi M_p^2 m}\right)^{2/3}e^{2\alpha(t)}h_{ij}\right], \tag{3}$$

where $a(t)=(2/2\pi M_p^2 m)^{1/3}e^{\alpha(t)}$ is the scale factor of the space section (3-sphere), and h_{ij} is the metric of the unit 3-sphere.

The action (3) includes the cosmological term λ. After standard calculations we obtain the Hamiltonian form of the action

$$S_H=\int dt(p_i\dot{q}^i-L)$$
$$=\int dtN\left[-\frac{e^{-3\alpha}}{2}p_\alpha^2+\frac{e^{-3\alpha}}{2}p_x^2+\frac{e^{-3\alpha}}{2x^2}p_\Theta^2-\frac{1}{2}\gamma e^\alpha+\frac{1}{2}e^{3\alpha}(x^2+\Lambda)\right], \tag{4}$$

where the canonical momenta are

$$p_\alpha=-\frac{e^{3\alpha}}{N}\dot{\alpha};\ p_x=\frac{e^{3\alpha}}{N}\dot{x};\ p_\Theta=\frac{e^{3\alpha}x^2}{N}\dot{\Theta}. \tag{5}$$

We introduce a dimensionless cosmological constant Λ and $\gamma=(3\pi M_p^2/2m^2)$.

Using the conservation of current j_μ of the field φ, we can exclude the constant momentum $p_\Theta=Q$, where Q is a new constant of the theory which plays a crucial role for the results. The final expression for the Hamiltonian action of the two-dimensional minisuperspace is (with the choice of lapse $N=1$):

$$S_H=\frac{1}{2}\int dt[e^{-3\alpha}(-p_\alpha^2+p_x^2)+m^2], \tag{6}$$

where we introduce the notation

$$m^2(\alpha,x)=\frac{e^{-3\alpha}Q^2}{x^2}-\gamma e^\alpha+e^{3\alpha}(x^2+\Lambda), \tag{7}$$

and the constraint equation is

$$H=e^{-3\alpha}(-p_\alpha^2+p_x^2)+m^2(\alpha,x)=0. \tag{8}$$

The Wheeler–DeWitt equation, which corresponds to Eq. (8) $[p_k\to(h/i)\nabla_k]$, is

$$\left[\hbar^2e^{-3\alpha}\left(\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial x^2}\right)+m^2(\alpha,x)\right]\Psi(\alpha,x)=0. \tag{9}$$

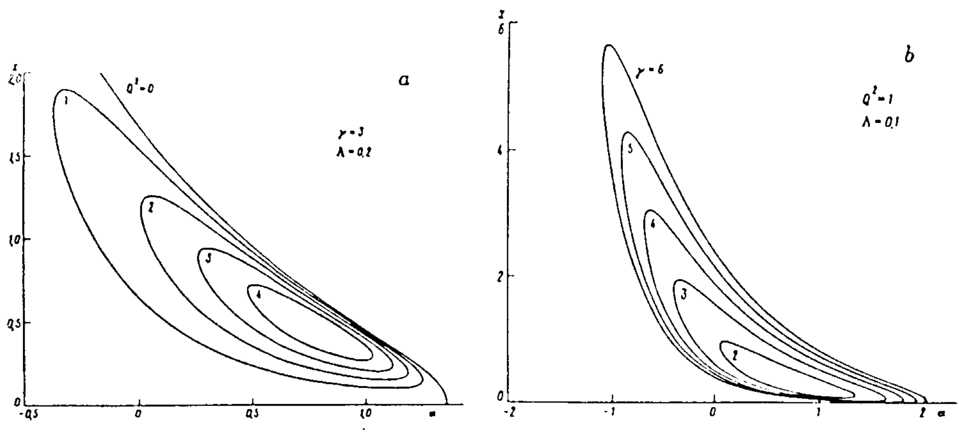

FIG. 1. The boundary $m^2=0$; (a) for fixed $\Lambda=0.2$ and $\gamma=3$ and different $Q^2=0, 1, 2, 3, 4$; (b) for fixed $\Lambda=0.1$ and $Q^2=1$ and different $\gamma=2, 3, 4, 5, 6$.

We write this WD equation in a simple form under the assumption that the metric of the minisuperspace is

$$g_{\mu\nu}=\begin{bmatrix} e^{3\alpha} & 0 \\ 0 & -e^{3\alpha} \end{bmatrix} \tag{10}$$

and its commutation brackets with a momentum p_α in the quasiclassical approximation is not important. The WD equation (9) which we obtained is a Klein–Gordon equation with m^2 which depends on the metric α and field x and which can be negative in some part of the minisuperspace. When we write the WD equation in the form (9), we assume that the p_Θ variable is frozen and that the corresponding motion is not quantized. Using the analogy between our variables, α and x, and the time and space coordinates, one possible interpretation is that in the region $m^2<0$ (Ref. 3) the processes of creation of particles are occurring in the second quantized theory of fields of universes $\Psi(\alpha,x)$. This region is classically forbidden. The shape of this region depends on three parameters: γ (of the mass of the scalar field), the constant Q, and the cosmological constant Λ. The possible shapes are shown in Fig. 1. Remarkably, we see that not only convex boundaries of the regions $m^2<0$, but also partly convex and partly concave boundaries can be obtained. The typical shape of the surface m^2 as a function of α and x and the corresponding equipotential map are shown in Fig. 2.

The constant Q is responsible for a closing of the upper part of the region $m^2<0$, and the cosmological constant Λ is responsible for a closing of the right part. The number of created universes and the most probable trajectories can be found from the solution of quasiclassical equations in classically allowed and forbidden regions.

1. Minkowski space—classically allowed region, $m^2>0$. The action in this case is

$$S_M=\frac{1}{2}\int dt[p^\mu p_\mu - m^2(\alpha,x)], \tag{11}$$

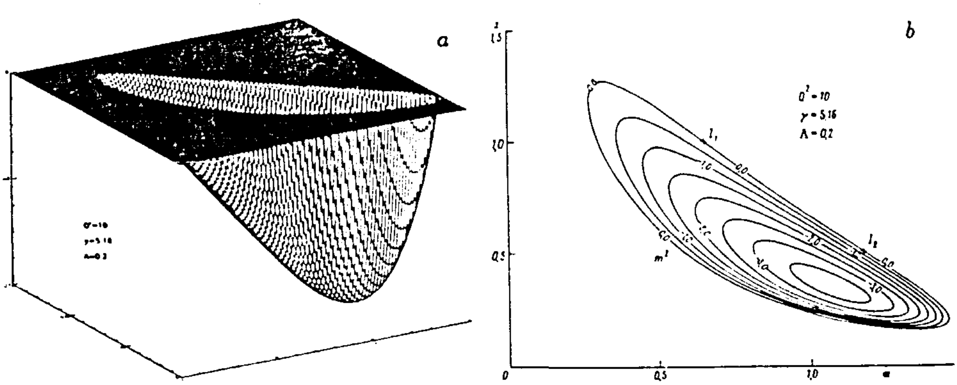

FIG. 2. The typical form (a) and the equipotential map (b) of the surface m^2 inside the Euclidean region ($\Lambda=0.2$, $Q^2=10$, $\gamma=5.1578$).

and the constraint equation is

$$H=\frac{1}{2}\left[p^{\mu}p_{\mu}+m^2(\alpha,x)\right]=0. \tag{12}$$

The corresponding Hamilton–Jacobi equation is

$$g^{\mu\nu}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\nu}}+m^2=0, \tag{13}$$

and

$$p^x=\dot{x}\quad g_{\mu\nu}=\begin{bmatrix} e^{3\alpha} & 0 \\ 0 & -e^{3\alpha}\end{bmatrix},\quad p_x=e^{3\alpha}\dot{x}$$

$$p^{\alpha}=\dot{\alpha},\qquad p_{\alpha}=-e^{3\alpha}\dot{\alpha}. \tag{14}$$

The variation of the action S_M

$$\delta S_M=-\frac{1}{2}\int dt\delta x^{\mu}\left[2\dot{p}_{\mu}+m^2(\log\sqrt{-g})\mu+\frac{\partial m^2}{\partial x^{\mu}}\right], \tag{15}$$

gives the equation for the characteristics

$$\dot{p}^{\mu}=-\frac{1}{2\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}(m^2\sqrt{-g}),\quad \sqrt{-g}=e^{3\alpha}. \tag{16}$$

This equation describes a dynamic four-dimensional system

$$\dot{x}=y,$$

$$\dot{\alpha}=z,$$

$$\dot{y}=-3yz-x+\frac{Q^2}{x^3e^{6\alpha}},$$

$$\dot{z}=-3z^2+3(x^2+\Lambda)-\frac{2\gamma}{e^{2\alpha}}, \tag{17}$$

with the first integral of motion

$$e^{3\alpha}(-z^2+y^2)+m^2=0. \tag{18}$$

2. Euclidean space—classically forbidden region, $m^2<0$. The Euclidean action S_E can be obtained by Wick rotation of the proper time $t\to -it$

$$S_E=\frac{1}{2}\int dt[p^\mu p_\mu+m^2(\alpha,x)] \tag{19}$$

with the constraint equation

$$H=\frac{1}{2}[p_\mu p^\mu-m^2(\alpha,x)]=0. \tag{20}$$

The variation of S_E,

$$\delta S_E=-\frac{1}{2}\int dt\delta x^\mu\left[-2\dot{p}_\mu+m^2(\log\sqrt{-g})_\mu+\frac{\partial m^2}{\partial x^\mu}\right], \tag{21}$$

gives the equation for the instanton

$$\dot{p}_\mu=-\frac{1}{2\sqrt{-g}}\frac{\partial}{\partial x^\mu}(m^2\sqrt{-g}),\quad \sqrt{-g}=e^{3\alpha}. \tag{22}$$

The Hamilton–Jacobi equation for the action of the instanton in a quasiclassical approximation ($I=S_E$) is

$$g^{\mu\nu}\frac{\partial I}{\partial\alpha^\mu}\frac{\partial I}{\partial\alpha^\nu}-m^2=0. \tag{23}$$

Its characteristics describe a dynamic four-dimensional system:

$$\dot{x}=y,$$

$$\dot{\alpha}=z,$$

$$\dot{y}=-3yz+x-\frac{Q^2}{x^3e^{6\alpha}},$$

$$\dot{z}=-3z^2-3(x^2+\Lambda)+\frac{2\gamma}{e^{2\alpha}}, \tag{24}$$

with the first integral of motion

$$e^{3\alpha}(z^2-y^2)+m^2=0. \tag{25}$$

In the Euclidean region it is necessary to find extremal instanton trajectories. The condition that the trajectory is extremal can be written as

$$\begin{cases} p^{\mu}_{(i,f)} p_{\mu(i,f)} = 0 \\ p^{\mu}_{(i,f)} e^{\mu} = 0 \end{cases}, \tag{26}$$

where i and j denote the initial and final points of the trajectory.

In (26) the first condition follows from the fact that the trajectory starts from the boundary $m^2=0$ and the second condition means that the action has an extremum with respect to variation of the initial and final points of the trajectory along the curve $m^2=0$. If e_μ is the tangential vector to the curve m^2-0 defined by

$$\epsilon_\mu \eta^\mu = 0, \tag{27}$$

then

$$n_\mu = \frac{\partial}{\partial x^\mu} (m^2) \,|_{m^2=0} \tag{28}$$

is the vector normal to the curve $m^2=0$. Since $\delta I_0/\delta x^\mu = p_\mu$, the extremum condition along the direction e_μ can be written

$$\frac{\delta I_0}{\delta x^\mu} e^\mu = p_{\mu(i,f)} e^\mu = 0, \tag{29}$$

consistent with the second equation in (26).

There are two possibilities of satisfying the condition (26), as was shown in Ref. 1. At the convex part of the boundary we can satisfy (26) only if $p_\mu = 0$. As we have shown, however, the boundary $m^2=0$ can have a concave part with inflection points. In the last case there is a possibility for the existence of a nontrivial solution which corresponds to the entrance of a trajectory into region $m^2<0$ with nonzero velocity along the tangent. Such a solution, if it exists, must satisfy the condition

$$p_\mu = \beta n_u, \tag{30}$$

where the parameter β defines the velocity at which the trajectory enters the region $m^2<0$. Since the vector p_μ at the entrance point is a light-like vector ($p_\mu p^\mu = 0$), the only possibility of satisfying this condition is to choose the parameters Q, y, and Λ in such a way that at one of the two inflection points the normal vector will be light-like; i.e.,

$$(n_\alpha)^2 - (n_x)^2 = 0. \tag{31}$$

In other words, it will have a slope of 45° (and the curve $m^2=0$ at this point will have a slope of $-46°$). At the inflection point we need to provide the possibility for the trajectory to enter and exit the region $m^2 \leqslant 0$. At the boundary of the Euclidean region, $m^2<0$, there are two singular points in Eq. (24). These points are defined by the conditions

$$y=0, \quad x - \frac{Q^2 e^{-6\alpha}}{x^3} = 0, \tag{32}$$

and they coincide with the leftmost and rightmost points of the region $m^2=0$, where

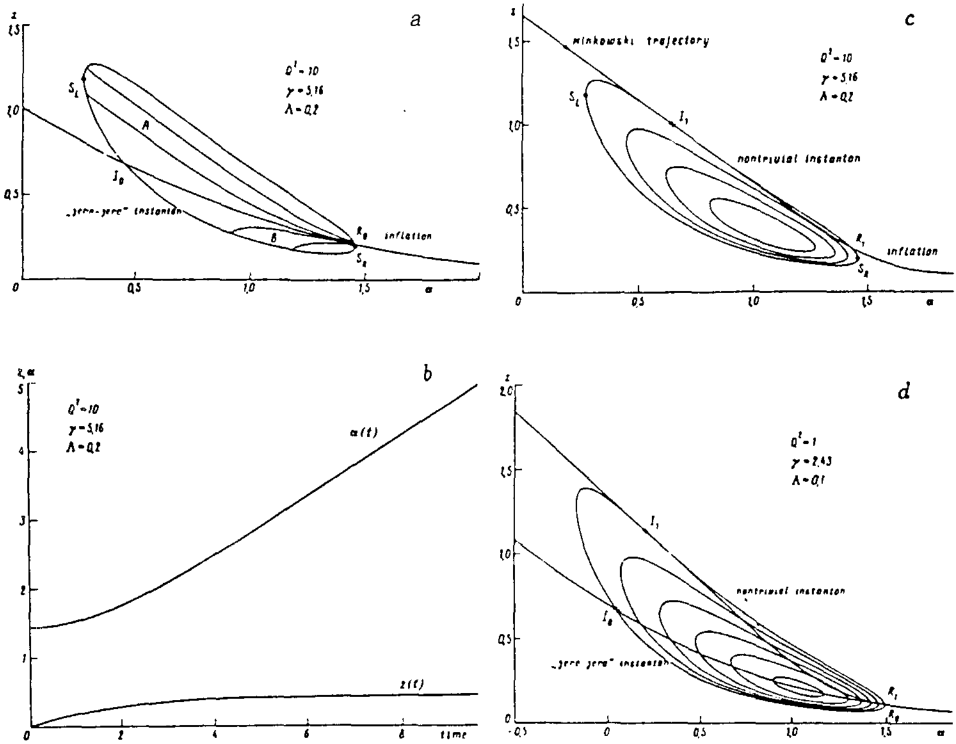

FIG. 3. The "zero–zero" instanton, the nontrivial instanton, the corresponding trajectories in the Minkowski space for $\Lambda=0.2$, $Q^2=10$, $\gamma=5.1578$ (a,c), $\Lambda=0.1$, $Q^2=1$, $\gamma=2.43$ (d) and the rate of the inflation (b).

$$\frac{\partial\alpha}{\partial x}=0,\quad n_x\frac{\partial m^2}{\partial x}=-\frac{2Q^2}{e^{3\alpha}x^3}+2xe^{3\alpha}=0. \tag{33}$$

The trajectories of the dynamic systems (17) and (24) in the Minkowski space and the Euclidean space naturally cannot be found analytically. We present the first results of numerical simulations. We start with the search for instantons in the region $m^2<0$. Selecting the parameter β in (30) which defines the initial velocity of the instanton in the minisuperspace, we find a trajectory starting at the inflection point I_1, which moves in the direction of the singularity S_R and reaches a point R_1, where the velocity $p_\mu=0$. After the reflection from this point it returns to the inflection point I_1, from which it can tunnel to the Minkowski space and leave the region $m^2<0$. At the reflection point R_1 which moves closer to the singularity S_R with decreasing Λ, the half-instanton can also tunnel and emit inflationary universe. All these possibilities are shown in Fig. 3a, where we show the Minkowski trajectory which describes the quantum oscillating universe tunneling into the Euclidean region, the trajectory of the instanton, and the inflationary universe leaving the Euclidean region. The corresponding rate of inflation is shown in Fig. 3b. The reflected trajectory from point R_1, which leaves the Euclidean region at the point I_1, describes in quasiclassical terms an unlimited contraction, i.e., a collapse. This trajectory asymptotically tends to knot at infinity with the slope $\dot{x} \simeq -\dot{\alpha}(p_\mu p^\mu \approx 0)$. The described picture has common features with the well-known process of quantum tunneling in quantum mechanics. Because of the large, absolute value of the negative action of the instanton, we will not describe here in detail the tunneling of the instanton. We hope to return to this topic later.

In addition to the described nontrivial instanton, there is a second instanton (the "zero–zero" instanton) with a trajectory which starts on the left convex part of the boundary at the point I_0 with $p_\mu = 0$ (Fig. 3c) and which propagates to the point $R_0(p_\mu = 0)$, where it can be either reflected or tunneled into the Minkowski space as an inflationary universe. The reflected half-instanton returns to the point I_0, where it can be reflected or tunneled into the Minkowski region. This trajectory propagates in the direction of the above-mentioned knot singularity at infinity. In the concave part of the boundary of the Euclidean "banana-like" region there are two inflection points, but only one of them has a normal which is a null vector. Therefore, inside this region we have simultaneously no more than two instantons: nontrivial or "zero–zero" instanton. The trajectories which enter from Minkowski region into the Euclidean region through the points I_1 and I_0 are emitted from the repulsive knot which is located at $z = +\infty$.

There is also another interesting possibility that the "zero–zero" instanton can be reflected many times from the points I_0 and R_0 and in some sense be trapped in the Euclidean region. Such an instanton can also be considered as a source of "pair" creation of an inflationary universe and antiinflationary (collapsing) universe from "nothing" (see Refs. 5 and 6).

Numerical analysis shows that the nontrivial instanton and the "zero–zero" instanton correspond to the local maxima of the action. Let us write the equation for the action of the instanton. From (19) and (20) we find

$$S_E = \int dt m^2 = \int dt p_\mu p^\mu = -\int_{\alpha_i}^{\alpha_f} d\alpha \sqrt{1 - \left(\frac{\partial x}{\partial \alpha}\right)^2} \sqrt{-m^2 e^{3\alpha}}. \tag{34}$$

The numerical calculation of the action based on this equation for the choice of the parameters Λ, γ, and Q, shown in Fig. 3d, gives $I_0 \simeq -7$ for the "zero–zero" instanton and $I_0 \simeq -4$ for the nontrivial instanton. The action of the de Sitter instanton in our notation is

$$I_{\rm dS} = -\frac{\gamma^{3/2}}{3\Lambda}. \tag{35}$$

The absolute value of this action is always larger than the corresponding action in the Euclidean region. It is well known that the action of the gravitational instanton is negative. Therefore,

$$|\Psi|^3 \propto e^{-2I}, \tag{36}$$

because an instanton is a large number, and because it is not possible to give its probabilistic interpretation. But if $|\Psi|^2$ is considered to be the number of created universes in the region of instability, $m^2 < 0$, that the Euclidean region can act as an amplifier, which selects only one of the trajectories from a large number arriving from the Minkowski region and makes the creation of the inflationary universe from the instanton most probable. This draws attention to the idea[3] that the ratio $e^{-2I_0}/e^{-2I_{dS}}$ can be interpreted as a probability for creation of the universe [for the definition of I_{dS} see (35)]. Further analysis will show how useful our proposed model is. In any case, it gives us a bridge between the quantum cosmology and the classical cosmology. If we consider the introduced parameters as world constants of a future theory, then our theory could be a model which would show how the initial conditions for the existing universe could follow from the theory and not be implemented *ad hoc.*

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[1)]The authors thank D. Brill for bringing to their attention the work of Lee.[4]
[2)]We expect that the introduction of interaction of the field of the type $\lambda\varphi^4$ or of a Higgs field will not change qualitatively the main results, but it needs special investigation.

[1] I. M. Khalatnikov and A. Mezhlumian, Phys. Lett. **A169**, 308 (1992).
[2] V. A. Belinsky, L. P. Grishchuk, Ya. B. Zel'dovich, and I. M. Khalatnikov, J. Exp. Theor. Phys. **89**, 346 (1985); V. A. Belinsky and I. M. Khalatnikov, J. Exp. Theor. Phys. **93**, 784 (1987).
[3] G. V. Lavrelashvili, V. A. Rubakov, M. S. Serebryakow, and P. G. Tinyakov, Nuclear Phys. **B329**, 98 (1990); V. A. Rubakov and P. G. Tinyakov, Nuclear Phys. **B342**, 430 (1990); W. Fishler, I. Klebanov, J. Polchinski, and L. Susskind, Nuclear Phys. **B327**, 157 (1989).
[4] K. Lee, Phys. Rev. Lett. **61**, 263 (1988).
[5] W. Hawking and I. G. Moss, Phys. Lett. **B110**, 35 (1982).
[6] A. D. Linde, JETP **60**, 211 (1984), Lett. Nuovo Cim. **39**, 401 (1984); Ya. B. Zeldovich and A. A. Starobinsky, Sov. Astron. Lett. **10**, 135 (1984); V. A. Rubakov, Phys. Lett. **B148**, 280 (1984); A. Vilenkin, Phys. Rev. **D30**, 549 (1984).

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