New multisoliton solutions of the Kadomtsev–Petviashvili equation

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A new class of exact rational solutions has been found for the Kadomtsev–Petviashvili equation which describes wave processes in media with a positive dispersion. These solutions represent spatially localized, steady-state multisolitons, which may be thought of as bound states of individual solitons.

1. There are two versions of the Kadomtsev–Petviashvili (KP) equation

$$(u_t+uu_x-\beta u_{xxx})_x=-u_{yy}, \tag{1}$$

which describe nonlinear wave processes in media with a positive dispersion ($\beta>0$; KP1) and a negative dispersion ($\beta<0$; KP2). Although formally both versions are completely integrable,[1] the problem of describing all physically significant solutions of these equations remains open, even for the steady state. The situation is particularly acute for KP1, which is considerably richer in terms of the description of various physical effects.

According to this equation, plane solitons are unstable with respect to self-focusing.[2] The spectrum of small perturbations is a decaying spectrum. This equation also allows the existence of two-dimensional solitons which are spatially localized in all directions.[3] Solutions of this sort were found first numerically[4] and later analytically.[5] It has been found that they have an algebraic structure (they can be expressed in terms of the ratio of two polynomials in x and y, and they fall off $\sim x^{-2}$, y^{-2} at infinity). They are stable with respect to small perturbations.[6] When they collide with each other, they may change in shape and may even experience phase shifts.[3,5]

It was later found (again by numerical calculations[7]) that KP1 has, in addition to single-soliton solutions, some bisoliton solutions, which may be thought of as bound states of two solitons in steady-state motion, one behind the other. The structure of such solutions has led to the conclusion that some even more-complex steady-state formations may exist: multisolitons.

In the present letter we find some exact analytic expressions which describe families of multisolitons. We show that they can be represented in the class of rational functions. A physical interpretation is offered in terms of bound states of individual solitons.

2. We seek steady-state solutions of Eq. (1) with $\beta>0$, assuming $u(x,y,t)=u(x+Vt,y)$, where $V>0$. Using the dimensionless variables $\xi=x\sqrt{V/\beta}$, $\eta=yV/\sqrt{B}$, $v=-u/V$, we rewrite (1) as

$$v_{\xi\xi}+v_{\eta\eta}=v_{\xi\xi\xi\xi}+(v^2)_{\xi\xi}/2. \quad (2)$$

We now use the Hirota substitution $v=12(\ln\varphi)_{\xi\xi}$, which puts Eq. (2) in bilinear form:[3,7]

$$\varphi(\varphi_{\xi\xi}+\varphi_{\eta\eta}-\varphi_{\xi\xi\xi\xi})=(\varphi_\xi)^2+(\varphi_\eta)^2+3(\varphi_{\xi\xi})^2-4\varphi_\xi\varphi_{\xi\xi\xi}. \quad (3)$$

This equation is known to have N-soliton solutions,[3,7] which can be written conveniently in the symmetric form

$$\varphi_N=\sum_\sigma \prod_{i=1}\sigma_i\exp\left(\frac{\sigma_i k_i\theta_i}{2}\right)\prod\prod_{1\leq i<j\leq N}a_{ij}(\sigma_i,\sigma_j), \quad (4)$$

where $\theta_i=\xi+p_i\eta+\theta_i^0$, $p_i^2=k_i^2-1$, $\sigma_i=\pm1$,

$$a_{ij}=\sqrt{(p_i-p_j)^2+3(\sigma_ik_i-\sigma_jk_j)^2}.$$

By virtue of the degeneracy with respect to the parameters k_i, we find real rational solutions from this formula. After we return to the variables $v(\xi,\eta)$, these solutions represent steady-state multisolitons. Here we must set $N=2M$, $p_{2n-1}=\bar{p}_{2n}$, $\theta_{2n-1}^0=\bar{\theta}_{2n}^0=\theta^0$, $n=1,\ldots,M$ and we must expand $\varphi_N(\xi,\eta)$ in a power series in k_i.

We will make use of the following properties[3] of the function φ_N:

- If $k_i=0$ $(i=1,\ldots,N)$, then $\varphi_N(\xi,\eta,\mathbf{k})=0$, where $\mathbf{k}=(k_1,\ldots,k_N)$;
- For $k_1=\pm k_{i+2j}$ we have $\varphi_N(\xi,\eta,\mathbf{k})=0$, $i=1,\ldots,N$, $j=1,\ldots,M-1$;
- $\varphi_N(\xi,\eta,\ldots k_i,\ldots,k_{i+2j}\ldots)=-\varphi_N(\xi,\eta,\ldots k_{i+2j},\ldots,k_i\ldots)+O(k^3)$.

With this choice of parameters, it follows from the properties of the function φ_N that the first few terms of the series vanish, leaving

$$\prod_{i=1}^N k_i\prod_{j=1}^{M-K}(k_i^2-k_{i+2j}^2)\left[\varphi_M(\xi,\eta)+\sum O(\mathbf{k})\psi(\xi,\eta)\right],$$

where $K=[\frac{i+1}{2}]$, $\varphi_M(\xi,\eta)$ is a polynomial which forms during the expansion of the exponential function, and $\Sigma O(\mathbf{k})\psi(\xi,\eta)$ is the remainder of the series. The degree of the polynomial $\varphi_M(\xi,\eta)$ can be determined easily from the following considerations. The coefficient in front of the square brackets is a homogeneous polynomial in k_i of degree $2M^2$. This polynomial arises in the series expansion of the matrix constants $a_{ij}(\mathbf{k})$ in (4) and of the exponential functions. When these conditions are taken into account, $M(M-1)$ constants of the type $a_{i,i+2j}$ are proportional to $k_i\pm k_{i+2j}$, and the expansion of the other matrix constants is described by $a_{ij}=2+O(k_i^2,k_j^2)$. The degree of the polynomial $\varphi_M(\xi,\eta)$ is thus found to be $P=2M^2-M(M-1)=M(M+1)$.

The entire coefficient in front of the square brackets can be omitted, since after we take logarithms and then differentiate with respect to ξ (in the spirit of the Hirota substitution), this coefficient vanishes. Furthermore, by taking the limit $\mathbf{k}\to0$ we can cause the remainder of the series to vanish also. The polynomial $\varphi_M(\xi,\eta)$, found as the limit of exact solution (4), must therefore also satisfy Eq. (3).

The first few polynomials found in this manner are written out explicitly below (their coefficients were found with the help of the Matematika software package on a personal computer).

$$M=1,\ P=2:\quad \varphi_1=\xi^2+\eta^2+3. \tag{5}$$

$$M=2,\ P=6:\quad \varphi_2=(\xi^2+\eta^2)^3+25\xi^4+90\xi^2\eta^2+17\eta^4-125\xi^2+475\eta^2+1875. \tag{6}$$

$M=3,\ P=12$:

$$\begin{aligned}\varphi_3=&(\xi^2+\eta^2)^6+2(\xi^2+\eta^2)^3(49\xi^4+198\xi^2\eta^2+29\eta^4)\\&+5(147\xi^8+3724\xi^6\eta^2+7490\xi^4\eta^4+7084\xi^2\eta^6+867\eta^8)\\&+\tfrac{140}{3}(539\xi^6+4725\xi^4\eta^2-315\xi^2\eta^4+5707\eta^6)\\&+\tfrac{1225}{9}(391314\xi^2-12705\xi^4+4158\xi^2\eta^2+40143\eta^4+736890\eta^2+717409).\end{aligned} \tag{7}$$

$M=4,\ P=20$:

$$\begin{aligned}\varphi_4=&(\xi^2+\eta^2)^{10}+30(\xi^2+\eta^2)^7(9\xi^4+38\xi^2\eta^2\\&+5\eta^4)+45(\xi^2+\eta^2)^5(369\xi^6+4275\xi^4\eta^2+5315\xi^2\eta^4+513\eta^6)\\&+5400(\xi^2+\eta^2)(65\xi^{12}+1902\xi^{10}\eta^2+8577\xi^8\eta^4+16476\xi^6\eta^6\\&+11531\xi^4\eta^8+5990\xi^2\eta^{10}+611\eta^{12})+3150(-5993\xi^{12}\\&+34138\xi^{10}\eta^2+340305\xi^8\eta^4+1214220\xi^6\eta^6+846825\xi^4\eta^8\\&+213178\xi^2\eta^{10}+114511\eta^{12}+664846\xi^{10}+1546350\xi^8\eta^2\\&+202020\xi^6\eta^4+14361060\xi^4\eta^6+9992910\xi^2\eta^8+6924974\eta^{10}\\&-56538105\xi^8-165596340\xi^6\eta^2+128842350\xi^4\eta^4+629226780\xi^2\eta^6\\&+381232335\eta^8)+6449625(1080248\xi^6+5161000\xi^4\eta^2\\&+14093560\xi^2\eta^4+6697992\eta^6+29548805\xi^4+134944810\xi^2\eta^2\\&+54062645\eta^4-107938610\xi^2+538716230\eta^2+917478185).\end{aligned} \tag{8}$$

After we go back to the original variables, the polynomial φ_1 gives us an individual soliton (Fig. 1a); this soliton was originally found in Ref. 5. The polynomial φ_2 corresponds to a two-humped solution, i.e., a bisoliton (Fig. 1b). This bisoliton had been constructed numerically in Ref. 7. The polynomials φ_3 and φ_4 lead to three- and four-humped multisolitons, respectively (Fig. 1, c and d). This procedure of constructing progressively more complex multistructures can evidently be continued even further, for $M=5,6,\ldots$, but the higher-order equations become progressively more complicated.

3. The steady-state multistructures constructed here can be interpreted in the theory of the interaction of solitons as classical particles.[8] According to that theory, if there are local minima of the field in the structure of solitons, one can expect the existence of bound states. In this case the solitons lie in the potential wells of each other. The one-dimensional version of this theory has been confirmed by numerical

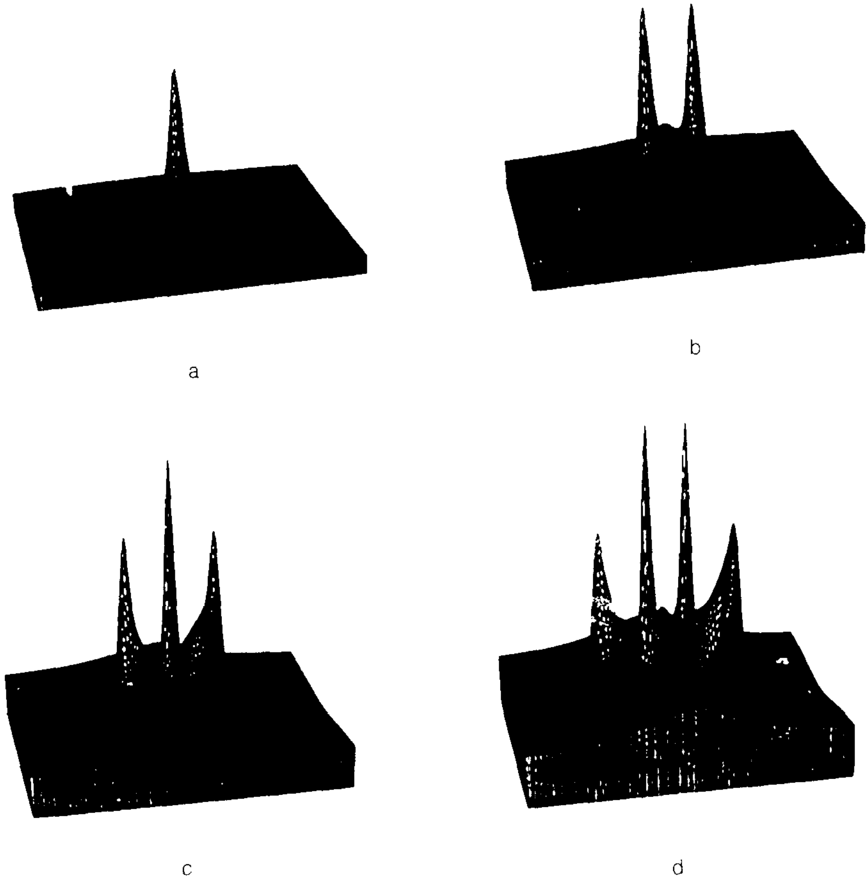

FIG. 1.

calculations and laboratory experiments.[8] In the two-dimensional case, this theory again yields good qualitative and even quantitative results, which make it possible to interpret the formal mathematical derivations in physical terms.

According to Ref. 8, solitons may be thought of as point particles which have generally different masses in the x and y directions. In particular, for KP1 the motion of two such particles is described by the Newton's equations

$$\hat{M}\frac{d^2\mathbf{r}}{dt^2}=\frac{\partial U}{\partial \mathbf{r}}, \tag{9}$$

where $\hat{M}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\mathbf{r}=(x,y)$, $U(\mathbf{r})=1/24\pi \int\int_{-\infty}^{\infty} v^2(\mathbf{r}')v(\mathbf{r}'+\mathbf{r})d\mathbf{r}'$, and we use the solution of Eq. (2) as $v(\mathbf{r})$.

Steady states evidently correspond to extrema of the potential function $U(\mathbf{r})$. Selecting the single-soliton solution in (5) as the simplest solution, and calculating

$U(\mathbf{r})$ with it, we easily find a unique local minimum at the point with coordinates (4.28, 0). The values of these coordinates agree well with the coordinates of the maximum of bisoliton (6): (4.62, 0). We could pursue this procedure to construct other multistructures and also to carry out calculations on time-varying interactions of multisolitons with each other.

There is an unfortunate error in Ref. 9. It led to the incorrect conclusion that no real nonsingular solutions of KP1, other than the simplest single-soliton solution, can exist in the class of rational functions.

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