Plasma relaxation in toroidal pinches and tokamaks

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A set of equilibrium states which a plasma may reach as the result of turbulent relaxation is found. These states correspond to solutions of the equation for an extremum of a Lyapunov functional. They are characterized by a reversal of the current at the periphery.

Woltjer[1] and Taylor[2] have concluded that a small-scale turbulence causes a plasma to relax to equilibrium with a relative minimum of the energy integral E, and then the turbulence decays. The conservation of the integral of the magnetic helicity, $\int hd^3x$, where $h=\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{B}=\text{curl}\,\mathbf{A}$, has been proposed as a condition for the minimum. Here **A** is the vector potential of the magnetic field **B**. The minimization occurs because the dissipation of energy is rapid in comparison with that of the helicity in the case of a short-wave turbulence. As a result, the relative energy minimum becomes an attractor. In the course of this relaxation, however, the plasma pressure p vanishes since there are no limitations on a minimum in terms of p.

A new first integral, $I_f=\int hf(p^{1/k}/h)d^3x$, which contains the pressure was proposed in Refs. 3 and 4 as a condition for a minimum of E and p. Here k is the adiabatic index of the plasma, and f is an arbitrary function. Under this more restrictive condition the plasma can relax to a state in which p has a significant maximum at the center of the plasma. In other words, a turbulent relaxation realizes a minimum of the Lyapunov functional $L=E+I_f$, where $E=\int[\rho v^2/2+p/(k-1)+B^2/8\pi]d^3x$, ρ is the density, and **v** is the plasma velocity.

These integrals were chosen because they are smooth with respect to the arguments **A**, p, **v**, and ρ, so the first and second variations of L are also smooth functionals. In the case of a short-wave turbulence, in the absence of external sources, the quantities v^2 and p relax most rapidly, because they are sensitive to the turbulent viscosity and to the thermal conductivity. The quantity $B^2=(\text{curl}\,\mathbf{A})^2$ relaxes more rapidly than $h=\mathbf{A}\cdot\text{curl}\,\mathbf{A}$, which satisfies a continuity equation[4] and which contains smaller derivatives. The ratio of the amplitudes of the corresponding fluctuations of these quantities is thus larger than a/λ, where a is a length scale of the plasma, and λ is a length scale of the turbulent fluctuations. As a result, if the plasma is in quasisteady state near a minimum of L, a comparatively rapid decrease in v^2, p, and B^2 sends the plasma into a state with a minimum of L, while there is almost no change in h. At the L minimum the plasma becomes stabler,[4,5] and the turbulence level should decrease. The extremum $\delta L=0$ leads to the relaxed-equilibrium equation[4]

$$\text{curl}\,\mathbf{B}=-2G\mathbf{B}-\nabla G\times\mathbf{A};\quad G=G(h);\quad \mathbf{B}=\text{curl}\,\mathbf{A}. \tag{1}$$

Here G is defined in terms of f. It gives the steady-state pressure as a function of the helicity:

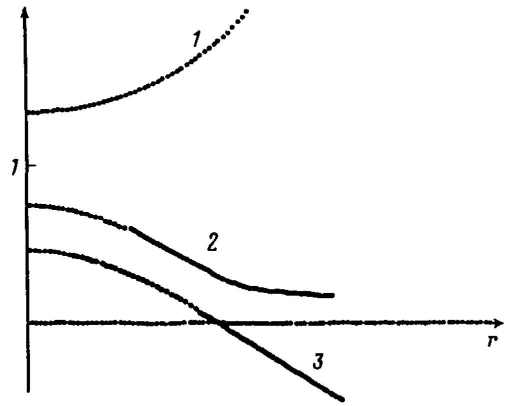

FIG. 1. Typical profiles of q in axisymmetric plasma confinement systems. *1*—Tokamaks; *2*—stabilized pinches; *3*—reversed-field pinches.

$$p(h)=\int h(\partial G/\partial h)dh/4\pi+\text{const.} \tag{2}$$

System of equations (1), (2) singles out a very limited subset of the set of plasma equilibria. The stability of such equilibrium states can be analyzed in the way that we would analyze the stability of ordinary equilibria, i.e., in the linear approximation.[5] In the case $G=\text{const}$, the current is parallel to B according to (1), and there is a force-free state. The corresponding equation has been solved analytically by Taylor.[2] We find other solutions of (1) numerically in a cylindrical geometry. For this purpose we assume that all quantities depend on the cylindrical radius r alone. At $r=0$ we set

$$A_z=\psi,\quad A_\phi=0,\quad B_z=1,\quad B_\phi=0,\quad \text{curl}_z\mathbf{B}=1,\quad G=-1/2. \tag{3}$$

Here ψ is a constant which, along with G, determines the structure of the equilibrium. Conditions (3) introduce units of magnetic field and length, B_z and $B_z/\text{curl}_z\mathbf{B}$, at the magnetic axis. It is convenient to replace $G(h)$ by a function of r: $G=g(r)/2$. Equation (2) then becomes

$$p=\int_a^r h(\partial g/\partial r)dr/8\pi,\quad \partial p/\partial r|_{r=a}=0. \tag{4}$$

We impose the further requirement that the plasma be cold and force-free at its boundary. These conditions can be satisfied by setting $\partial g/\partial r|_{r=a}=0$ in systems in which h^2 increases toward the boundary [in solutions of (1) of the tokamak type] or in which we have $h|_{r=a}=0$ (in pinches). To cause the current to vanish at the boundary of the tokamak-like solution of (1), we should also require $g|_{r=a}=0$. As the boundary is approached, g thus vanishes more rapidly than $\partial g/\partial r$ does. It can be seen from (1) that the result is a change in the sign of the toroidal current in solutions of this sort.

By specifying various values of g and ψ we can find solutions of (1) in the form of three fundamental axisymmetric configurations which are encountered experimentally: a tokamak, a stabilized pinch, and a reversed-field pinch. The primary distinction among the three is the profile of the safety factor q (Fig. 1). A stabilized-pinch

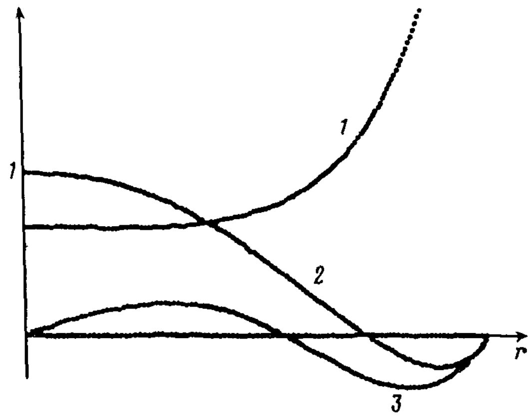

FIG. 2. Profiles of (1) q, (2) $\text{curl}_z\mathbf{B}$, and (3) $\text{curl}_\phi\mathbf{B}$ of relaxed equilibria in a tokamak with a convex pressure profile at the center, in the dimensionless units of (3). The condition $q=8.6$ holds at the boundary.

solution of (1) has been studied in Ref. 4 and other papers. Solutions of (1) in the form of a reversed-field pinch have a high current density at the boundary, which is not altogether convenient for fusion purposes.

Let us consider tokamak solutions of (1) in the case in which the pressure has a parabolic profile at the center. In the simplest case we would have $g=-[1-(r/a)^2]^2$. In this case the profiles of the magnetic field and the pressure differ only slightly from their usual profiles. Figure 2 shows a profile $q=rB_z/RB_\phi$ for $\psi=-0.2$, $a=1$, and $R=1.5$. There is a steep rise near the boundary. This rise is caused by the change in the sign of the toroidal current in this region; this change in sign leads to a partial screening of the poloidal field (Fig. 2). At the axis we have $q=2/R$ and $h=\psi$ according to (3). The ratio of the pressure to the magnetic pressure at the axis is $\beta=0.22$ in this example. The value of β increases with decreasing ψ.

Let us consider a flatter pressure profile—a cubic parabola—near the axis. This would be the case with, for example, $g=-[1-(r/a)^3]^2$. Near the axis we have a force-free configuration, in which q falls off with distance from the axis according to the analytic solution of Ref. 2. Figure 3 shows the results of a solution of (1) with

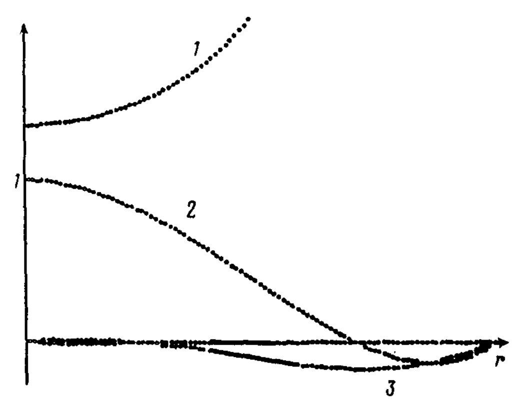

FIG. 3. Profiles of (1) q, (2) $\text{curl}_z\mathbf{B}$, and (3) $\text{curl}_\phi\mathbf{B}$ of relaxed equilibria in a tokamak with a flat pressure profile at the center. The condition $q=3.5$ holds at the boundary, while q has a minimum at $r=0.4$.

$\psi = -0.4$, $a=2$, and $R=3$. Here q has a slight minimum at $r=0.4$. There is also a more pronounced reversal of the current. At the center we have a paramagnetism, and at the edge of the plasma we have a diamagnetism. At the axis we have $\beta=0.47$. Such a profile looks unstable. However, the experiments of Ref. 6 yield a q profile similar to this (when we allow for the experimental errors). Under the conditions described in Ref. 6, the equilibrium was force-free, at both the edge and center of the plasma. The q profile was thus not monotonic.

Consequently, the circumstance that, according to Holm *et al.*,[5] the stability of the equilibrium corresponding to an extremum of the Lyapunov functional can be analyzed by a linearization method means that we can extend relaxation theory to the study of tokamaks. It thus becomes possible to dramatically reduce the uncertainty in the choice of the best equilibrium parameters. In relaxed states of this sort, described by Eq. (1), the field is force-free at the edge, as is seen in a change in the sign of the current and in a steep increase in q. According to Ref. 7, where the conditions for a kink instability were found, the presence of a region with a negative current (more precisely, a region with a positive current gradient near a boundary) substantially improves the stability.

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