Vortex motion in Fermi superfluids and the Callan–Harvey effect

G. E. Volovik
Landau Institute for Theoretical Physics, 117334, Moscow, Russia and Centre de Recherches sur les Tres Basses Temperatures, 38042 Grenoble, France

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The spectral flow along the anomalous branches of the fermions, which is localized in the core of a quantized vortex in Fermi superfluids, defines at low temperatures the parameter D' which characterizes the reactive force between the vortex and the system of normal fermions. This reversible momentum exchange between coherent condensate motion in three dimensions and the one-dimensional motion of the localized fermions is equivalent to the Callan–Harvey process of the anomaly cancellation. The number of anomalous branches is related to the vortex winding number.

1. Introduction. In axially symmetric quantized vortices in Fermi superfluids the energy levels $E(n_r,n_l,k_z)$ of the fermionic quasiparticles, which are localized in the vicinity of the vortex core, are characterized by the quantum numbers appropriate for the axial symmetry: the linear momentum along the vortex axis, k_z, the orbital angular momentum, n_l, and the radial quantum number, n_r. The distance between the levels with different n_r, and that between the levels with different n_l at given n_r are characterized by two energy scales, ω_τ and ω_l, respectively. In the singular vortex, ω_τ is on the order of the gap parameter $\Delta \sim T_c$, while $\omega_l \sim \Delta^2/E_F \ll \Delta$ (Ref. 1). The vortex dynamics and thermodynamics are essentially different in two regions of low temperatures; $\Delta \gg T \gg \omega_l$ and $T < \omega_l$.

We consider here the first region, which contains the anomalous (chiral) branches of the localized fermions, corresponding to $n_r=0$. The spectrum $E(n_r=0,n_l,k_z)$ forms the band with nearly equidistant levels, $E(n_r=0,n_l,k_z) \approx \omega_l(k_z)n_l$, which as a function of (discrete) n_l crosses the zero energy and thus produces the finite density of states $N(\omega)$ if $\omega \gg \omega_l$. We show that in this region there are several general topological features in the vortex dynamics produced by the effect of spectral flow. This effect, defined by the winding number of the vortex, does not depend on behavior of the spectrum in the region below ω_l, where, depending on the vortex core structure, the spectrum $E(n_r,n_l,k_z)$ may or may not cross zero as a function of k_z (Refs. 2 and 3).

We assume that either ω or the energy level width τ^{-1} is larger than the distance ω_l between the n_l levels, so the energy spectrum can be assumed to be a continuous function of n_l. The main result is that the spectral flow along the chiral branch of n_l defines the reactive parameter D', which enters the force balance for the moving vortex:

$$\rho_s(\mathbf{v}_s-\mathbf{v}_L)\times\mathbf{k}+D(\mathbf{v}_n-\mathbf{v}_L)-D'(\mathbf{v}_n-\mathbf{v}_L)\times\mathbf{k}=0. \tag{1.1}$$

Here $\mathbf{k}$ is the circulation vector, $k = n\pi\hbar/m$, where n is the vortex winding number (the number of circulation quanta), and m is the bare mass of the fermion.

The first term is the conventional Magnus force which arises when the velocity $\mathbf{v}_L$ of the vortex is different from the superfluid velocity $\mathbf{v}_s$; ρ_s is the superfluid density far from the vortex, which is close to the total density ρ since $T \ll T_c$. This force comes from the flux of the linear momentum from the vortex to infinity.

The second term describes the friction force acting on the vortex from the normal component, while the last term is the reactive force which arises when $\mathbf{v}_L$ deviates from the normal velocity $\mathbf{v}_n$. The parameter D' is assumed to be proportional to the normal density ρ_n. However, as was shown in Ref. 4 for a singular vortex, while ρ_n disappears at $T \ll T_c$, the parameter D' does not: it approaches finite value close to ρ provided that either $\omega \gg \omega_l$ or $\tau^{-1} \gg \omega_l$.[5,6] The same result was obtained for the continuous vortices in a superfluid ^{3}He–A. Here we wish to stress that for all the vortices, singular or continuous, the parameter D' at $\omega_l \ll T \ll T_c$ is defined by the same mechanism of the momentum transfer due to the level flow.

The process of the momentum transfer from the superfluid vacuum to the normal motion of fermions within the core is the realization of the Callan–Harvey effect[8] for vortices in condensed matter: the anomalies—nonconservation of linear momentum in the one-dimensional world of the vortex core fermions and in the three-dimensional Bose condensate outside the vortex core—compensate for each other. This is the same kind of Callan–Harvey effect which was discussed earlier for the motion of arbitrary textures[9,10] in ^{3}He–A. ^{3}He–A is, however, very specific superfluid, since it always contains due to its internal topology the gap nodes in the spectrum.[7] The nodes account for the momentum nonconservation if one considers the superfluid condensate motion alone. This is the result of transfer of the momentum to the normal fermionic system due to the level flow through the gap nodes. In contrast with ^{3}He–A, where the gap nodes are always present, the Callan–Harvey effect for vortices occurs in any Fermi superfluid: The anomalous fermionic n_l branch which mediates the momentum exchange always appears in the singular or continuous core, because of the nontrivial topology of the quantized vortex.

2. Anomalous branch of localized fermions. Since the result does not depend on the particular features, and since it is completely defined by the topology, we consider here a simple and well-known case of an axisymmetric singular vortex in a superfluid or in a superconductor with an s-wave pairing. The orbital number n_l is considered as the continuous variable, so one can use the quasiclassical approximation for the fermions localized in the vortex core. The Bogolyubov Hamiltonian for the fermions with a given spin projection is a 2×2 matrix:

$$\mathbf{H} = \hat{\tau}_3 \mathbf{q} \cdot (-i\nabla)/m + \hat{\tau}_1 \mathrm{Re}\Delta(\mathbf{r}) - \hat{\tau}_2\ \mathrm{Im}\ \Delta(\mathbf{r}). \tag{2.1}$$

Here $\mathbf{q}$ is the quasiparticle momentum in the transverse plane, and $\Delta(\mathbf{r}) = e^{in\phi}|\Delta(r)|$ is the gap function in an axisymmetric vortex with the winding number n.

The quantum numbers which characterize the fermionic levels in this approximation are (i) the transverse momentum of the quasiparticle q, which is related to the longitudinal projection of the momentum $q^2 = k_F^2 - k_z^2$, (ii) the radial quantum number

n_r, and (iii) the continuous impact parameter $\rho=\hat{z}\cdot(\mathbf{r}\times\mathbf{q})/q$. It is related to the angular momentum $\hbar n_l$ as $\hbar n_l=q\rho$. Introducing the coordinate $x=\mathbf{r}\cdot\mathbf{q}/q$ along $\mathbf{q}$, such that $r^2=\rho^2+x^2$, and assuming that in the important regions we have $|\rho|\ll|x|$, we obtain the dependence of the gap function in the singly quantized vortex ($n=1$) on x and ρ:

$$\Delta(\mathbf{r})\approx|\Delta(|x|)|\left[\mathrm{sign}(x)-i\frac{\rho}{|x|}\right], \tag{2.2}$$

and the Hamiltonian:

$$\mathbf{H}=\mathbf{H}^{(0)}+\mathbf{H}^{(1)},\quad \mathbf{H}^{(0)}=-i\hat{\tau}_3\frac{q}{m}\nabla_x+\hat{\tau}_1|\Delta(|x|)|\mathrm{sign}(x),\quad \mathbf{H}^{(1)}=\hat{\tau}_2\rho\frac{|\Delta(|x|)|}{|x|}. \tag{2.3}$$

The Hamiltonian $\mathbf{H}^{(0)}$ is supersymmetric and has a zero eigenvalue with the eigenfunction:

$$\Psi^{(0)}\propto(1-\hat{\tau}_2)\exp-\frac{m}{q}\int_0^{|x|}dy|\Delta(y)|. \tag{2.4}$$

Using the first order in perturbation $\mathbf{H}^{(1)}$, we obtain the lowest energy levels:

$$E(n_r=0,n_l,k_z)\approx<0|\mathbf{H}^{(1)}|0>=-\rho<\frac{|\Delta(|x|)|}{|x|}>=-n_l\omega_l(q),$$
$$\omega_l(q)=\frac{1}{q}\frac{\int_0^\infty dx|\Psi^{(0)}(x)|^2|\Delta(x)|/x}{\int_0^\infty dx|\Psi^{(0)}(x)|^2}. \tag{2.5}$$

This is the anomalous branch of the low-energy localized fermions obtained in Ref. 1. If the energy spectrum is assumed to be a continuous function of n_l, this anomalous branch will cross zero at $n_l=0$. It is shown in Sec. 4 that the number of such anomalous branches N_{zm}, is completely defined by the number n of circulation quanta ($N_{\mathrm{zm}}=2n$; for $n=1$ two branches correspond to two spin projections). A similar relationship between the number of fermionic zero modes in the core of the string and the string winding number n is used in the relativistic field theories.[11,12] The difference is that in the core of the string the zero modes are exact, while in the condensed matter vortices they are approximate on the scale $\omega\gg\omega_l$.

3. Spectral flow and the mutual friction parameter D′. Let us now consider a vortex moving with the velocity $\mathbf{v}_L$ relative to the heat bath. In this case the coordinate $\mathbf{r}$ is replaced by $\mathbf{r}-\mathbf{v}_Lt$ and the parameter ρ, which is included in the quasiparticle energy in Eq. (2.5), is shifted with time:

$$E(n_r=0,n_l,k_z,t)=-\left(\rho-\frac{\epsilon(\mathbf{q})}{q}t\right)q\omega_l(q)=-(n_l-\epsilon(\mathbf{q})t)\omega_l(q), \tag{3.1}$$

where $\epsilon(\mathbf{q})=\hat{z}\cdot(\mathbf{v}_L\times\mathbf{q})$ acts on the fermions localized in the core in a similar way as the electric field acts on the fermions localized on a string in the relativistic quantum theory. In this field the number of fermionic levels which cross zero per unit time is

$$\partial_t n_l = \epsilon(\mathbf{q}) = \hat{z} \cdot (\mathbf{v}_L \times \mathbf{q}). \tag{3.2}$$

The rate of the quasiparticle momentum transferred from the vacuum (from the levels below zero) along the anomalous branch is

$$\partial_t \mathbf{P} = \Sigma \mathbf{q} \partial_t n_l(\mathbf{q}) = \frac{1}{2} N_{\text{zm}} \int_{-k_F}^{k_F} \frac{dk_z}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \mathbf{q} \epsilon(\mathbf{q}) = \pi n \frac{k_F^3}{3\pi^2} \hat{z} \times \mathbf{v}_L . \tag{3.3}$$

The factor $\frac{1}{2}$ compensates for the double counting of the particles and holes. This gives the D' parameter in the force acting from the normal component if the vortex moves with respect to the normal heat bath:

$$D' = m \frac{k_F^3}{3\pi^2}. \tag{3.4}$$

Here it is implied that all the quasiparticles created from the negative levels of the vacuum state finally become the part of the normal component; i.e., there is a nearly reversible transfer of the linear momentum from the fermions to the heat bath. This should be valid in the limit of a large scattering rate: $\omega_l \tau \ll 1$. The small retardation in this process leads to the effective friction force, $D \propto \omega_l \tau D'$ (see Refs. 3 and 6).

Note that D' is very close to the density ρ, but it is not exactly equal to ρ. The mass density is $m(k_F^3\, 3\pi^2)$ only in the normal Fermi liquid, while in the superfluids $\rho \neq m(k_F^3/3\pi^2)$, but is close to this value if $\Delta \ll E_F$. For the singular vortex in s-paired superfluids D' therefore corresponds to the density at the vortex axis, where the gap disappears: $D' = \rho(r=0)$. In a continuous ^{3}He–A vortex, D' coincides with the C_0 parameter, which is responsible for the chiral anomaly, $D' = C_0 = m(k_F^3/3\pi^2)$ (Ref. 5).

4. Number of anomalous branches and vortex winding number. The general relation between N_{zm} and n can be found using the Green's functions (in the spirit of Refs. 2 and 7). Here we consider the simplified derivation, which uses 2×2 Bogolyubov matrix (2.1). The number of anomalous branches of the spectrum $E(\rho)$, which cross zero as a function of ρ, coincides with the number of the topological zeros of the classical energy[2,7] $E(\rho, x, p_x)$. The classical limit of Hamiltonian (2.1) is expressed in terms of the vector function $\mathbf{m}(\mathbf{s})$ in the 3D space of the parameters $\mathbf{s} = (\rho, x, p_x)$:

$$\hat{H}_{\text{class}}(\mathbf{s}) = \boldsymbol{\tau} \cdot \mathbf{m}(\mathbf{s}). \tag{4.1}$$

The components of $\mathbf{m}(\mathbf{s})$ are

$$m_3(\mathbf{s}) = q p_x / m, \quad m_1(\mathbf{s}) = \text{Re } \Delta(\rho, x), \quad m_2(\mathbf{s}) = -\text{Im } \Delta(\rho, x). \tag{4.2}$$

The number of zeros of this vector function [the points $\mathbf{s}_0$ where $\mathbf{m}(\mathbf{s}_0) = 0$ and therefore $E(\mathbf{s}_0) = 0$] and thus the number of anomalous branches are given by the topological invariant:[7]

$$N_{\text{zm}} = \frac{1}{8\pi} \int_\sigma dS^i e_{ikl} |\mathbf{m}(\mathbf{s})|^{-3} \left(\mathbf{m} \cdot \frac{\partial \mathbf{m}}{\partial s_k} \times \frac{\partial \mathbf{m}}{\partial s_l} \right), \tag{4.3}$$

where the integral is over the closed surface σ about zeros. For the gap function $\Delta(\mathbf{r}) = e^{in\phi} |\Delta(r)|$ in the vortex with the winding number n we obtain $N_{\text{zm}} = n$ which should be multiplied by two if one takes into account two spin projections. The general relation which does not depend on the detailed structure of the vortex therefore is

$$N_{zm}=2n. \tag{4.4}$$

Note that the quantity N_{zm} changes sign for vortices with a negative winding number. This occurs because N_{zm} is an algebraic quantity, since it shows that $E(n_l)$ increases or decreases upon crossing zero energy.

5. Conclusion. The Magnus force acting on the vortex from the superfluid motion results from the reversible flux of the linear momentum from the vortex to infinity. In contrast, the reactive force from the normal component is attributable to the reversible flux of the momentum from the vortex to the region near the axis, i.e., to the core region. Within the core, the linear momentum of the vortex transforms to the linear momentum of the fermions when the fermionic levels on the anomalous branches cross the chemical potential. This type of the Callan–Harvey effect does not depend on the detailed structure of the vortex core or even on the type of pairing. It is the same for the singular and continuous vortices. The topological result (4.4) for the number of anomalous branches, which as functions of the impact parameter ρ cross zero energy, remains valid for vortices in the p-wave superfluids, e.g., in ^{3}He–A and ^{3}He–B. In the case of singly quantized vortices in ^{3}He–B two anomalous branches have been obtained by Schopohl.[13] While for the most symmetric vortex they cross zero at $\rho=0$, as in Eq. (2.5), for the vortex with a broken symmetry in the core the crossing occurs at finite ρ. This circumstance, however, does not change Eq. (3.2) for the spectral flow or the D' value (3.4).

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