Universality in quantum chaotic spectra

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The response of the energy levels of a quantum chaotic system to an arbitrary external perturbation has been studied. The statistical properties of the energy dispersion were found to depend only on the mean-level spacing and on the generalized conductance. A new rescaling has been introduced, after which the statistical correlations of the energy levels become universal. Evidence is provided from both analytical and numerical calculations.

The Wigner-Dyson distribution[1] has shown considerable success in describing the level correlations of a variety of complex systems ranging from systems with many degrees of freedom and strong interactions (such as atomic nuclei) to the quantum-mechanical motion of particles in irregular potentials (such as disordered metallic grains or quantum dots). It is equally capable of describing Hamiltonians governed by simple dynamics such as hydrogen in an external magnetic field. In this sense, the distribution is a manifestation, if not a definition, of quantum chaos. However, frequently we are interested in the response of the energy levels of a system to the action of an external perturbation.[2–4] Here we introduce a simple rescaling which reveals a higher level of universality in spectral correlations.

Let us assume that a Hamiltonian $\mathscr{H}$ depends on an external perturbation through a parameter X which has eigenvalues given by the random functions $E_i(X)$. Without loss of generality we assume that $\langle \partial E_i(X)/\partial X\rangle = 0$, where $\langle \cdots \rangle$ denotes an average over X and over a typical range of levels. We demonstrate that the rescaling,

$$x = \sqrt{C(0)}X, \quad \epsilon_i(x) = E_i(X)/\Delta, \tag{1}$$

where Δ is the mean-level spacing, and

$$C(0) = \left\langle \left(\frac{\partial \epsilon_i(X)}{\partial X} \right)^2 \right\rangle , \tag{2}$$

makes the statistics of $\epsilon_i(x)$ universal, which depend only on the Dyson ensemble. $C(0)$ describes the sensitivity of the spectrum to variations in X and provides, in addition to Δ, the only characteristic of the system.

The energy dissipation rate $\partial \epsilon/\partial t$ caused by a time-dependent perturbation $X(t)$,

$$\frac{\partial \epsilon}{\partial t} = \frac{\beta \pi}{2} \hbar C(0) \left(\frac{\partial X}{\partial t} \right)^2 , \tag{3}$$

where $\beta = 1(2)$ denotes the Dyson orthogonal (unitary) ensemble, gives $C(0)$ the physical meaning of "conductance." An analogous formula was proposed by Wilkinson[2] by making reasonable assumptions in random matrix theory. In fact, for disor-

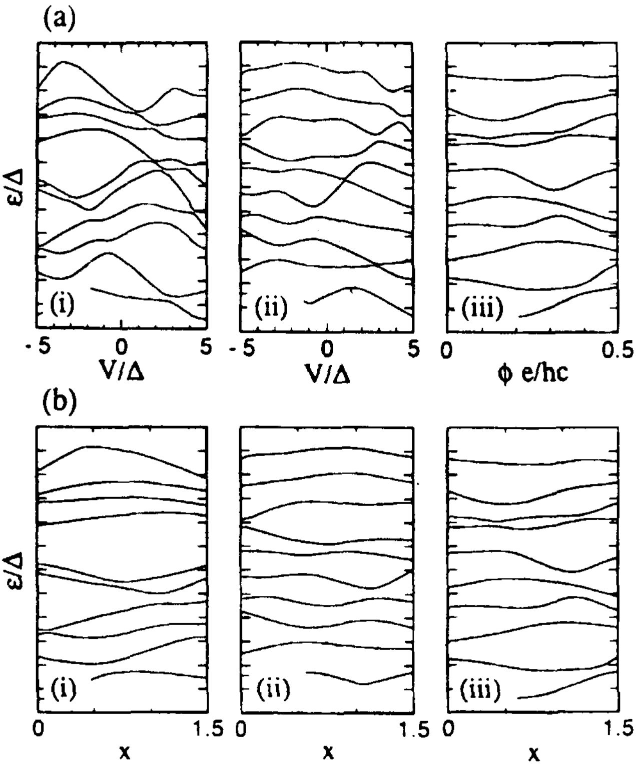

FIG. 1. (a) Bare and (b) rescaled spectra shown for a typical range of energy levels for (i) $X\equiv V/E_c$, (ii) $X\equiv V/E_c$, and (iii) $X\equiv \phi e/hc$. In (i) and (ii) the scattering is from impurities with $W=2.4$, while in (iii) it is by an irregular boundary with the geometry shown in the inset in Fig. 2. Case (ii) differs from (i) in that an applied magnetic field breaks T-invariance, making the symmetry of (ii) unitary. After rescaling the unitary samples, (ii) and (iii) become statistically equivalent and distinct from the orthogonal sample, (i).

dered systems Eq. (3) can be derived exactly.[5] Although Eq. (3) has the form of a fluctuation-dissipation theorem, we note that $C(0)$ represents mesoscopic (sample to sample or Fermi level to Fermi level), rather than thermal, fluctuations. The relation (3) can therefore be described as a mesoscopic fluctuation-dissipation theorem.

The universality can be illustrated by examining the chaotic motion of a particle scattering from either a disordered array of impurities (weakly disordered metals) or from an irregular boundary (billiard). In the former, after averaging over the realizations of disorder, the rescaling can be demonstrated rigorously. The details are presented elsewhere.[5] The results of a numerical simulation presented below show that ensemble averaging is not crucial. Averaging over a typical range of energy is sufficient to ensure universality.

First, we can apply an external perturbation in the form of an Aharonov–Bohm flux through a ring. As particles circulate around the ring, the wave function acquires a phase, $2\pi\phi e/hc\equiv 2\pi X$. Substitution of (2) in (3) gives the Thouless formula[6] for

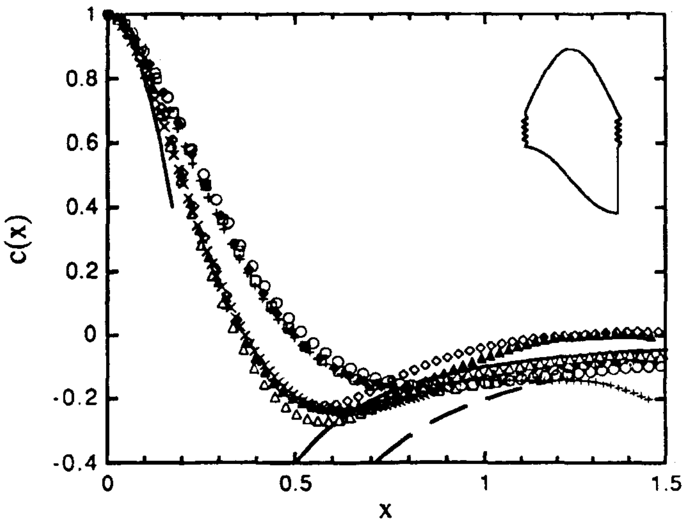

FIG. 2. The $c(x)$ measured from disordered and chaotic samples for several cases. Measurements with $X \equiv V/E_c$ are shown for a range of disorder; $W = 1.9$ (open circles), $W = 2.4$ (filled diamonds), and $W = 2.9$ (open squares) at zero magnetic field; $W = 2.4$ (open diamonds) at nonzero field; and for a chaotic billiard at zero field (crosses $+$), and at nonzero field (crosses $\times$). Measurements with $X \equiv \phi e/hc$ are shown for disorder $W = 2.4$ (filled triangles), and for a chaotic billiard (open triangles). The chaotic billiard (shown in the inset) is assumed to be connected along the zigzag edge. The asymptotic approximations to $c(z)$ are shown for the unitary (continuous) and orthogonal (broken) ensembles. All measurements from samples with impurity scattering are averaged over four realizations of the disorder.

conductance[5] $G = e^2 C(0)/2h$. This formula has the form which was recently proposed in Ref. 7.

The change in sign of ϕ under T-reversal implies unitary symmetry. We will also examine a second external perturbation which can act on a system taken from either an orthogonal or unitary ensemble. Applying a potential step across a sample (with half the sites raised and half lowered by a potential V) and setting $X \equiv V/E_c$, we obtain[5] $G = 12\pi^2 e^2 C(0)/h$. A fixed magnetic field can be used to drive the system from an orthogonal to a unitary ensemble [reducing $C(0)$ by a factor of 2]. According to universality, after rescaling with (1) the spectra should become statistically indistinguishable from other unitary ensembles. Comparison with $X \equiv \phi e/hc$ therefore provides a critical test of the universality.

The simulations were performed using a tight-binding Anderson model with on-site energies chosen randomly from the range $-W/2 < W_i < W/2$ ($W = 0$ for the billiard). The universality is illustrated qualitatively in Fig. 1, where the rescaling is applied to three different spectra. A quantitative test of the rescaling is provided by the autocorrelation function of level "velocities,"

$$c(x) = \left\langle \frac{\partial \epsilon_i(\bar{x}+x)}{\partial \bar{x}} \frac{\partial \epsilon_i(\bar{x})}{\partial \bar{x}} \right\rangle, \tag{4}$$

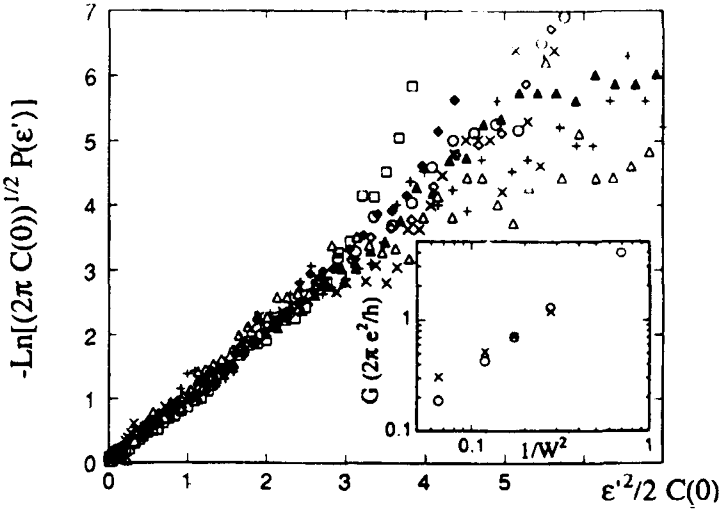

FIG. 3. Measurements of the velocity distribution function $P(\epsilon')$, where $\epsilon' \equiv \partial\epsilon/\partial x$, with symbols corresponding to those used in Fig. 2. The variation of conductance G as a function of disorder $1/W^2$ is shown in the inset, with $X \equiv \phi e/hc$ (open circles), and $X \equiv V/E_c$ (crosses X). All measurements for impurity scattering are averaged over four realizations of the disorder.

shown in Fig. 2. All the data from the billiard and disordered samples collapse on one of the two curves according to the Dyson ensemble. In particular, $c(x)$ measured with $X \equiv V/E_c$ at zero magnetic field collapses onto the "orthogonal curve," while at nonzero field it follows the "unitary curve," which coincides with the flux autocorrelator $(X \equiv \phi e/hc)$.

We have not succeeded in evaluating $c(x)$ analytically, and the behavior of the function is available only in the asymptotic region of large x, where[4,8,5] $c(x) = -2/\beta\pi^2 x^2$, and for the unitary ensemble, $c(x) = 1 - 2\pi^2 x^2 + O(x^4)$ at vanishing x. These results can be obtained from the autocorrelator of the density of states fluctuations,

$$k(\omega,x) = \langle (1/v^2) \sum_{ij} \delta[\epsilon - \epsilon_i(\bar{x})]\delta[\epsilon - \omega - \epsilon_j(x + \bar{x})] \rangle - 1,$$

which have been determined exactly for orthogonal and unitary ensembles.[5] For example, in the unitary ensemble,

$$k_u(\omega,x) = \text{Re} \int_{\pi}^{\infty} d\lambda_1 \int_{-\pi}^{\pi} d\lambda \frac{1}{2\pi^2} \exp[x^2(\lambda^2 - \lambda_1^2) + i\omega(\lambda - \lambda_1)]. \quad (5)$$

The expression $k(\omega,x)$ can be used to show that the distribution of level velocities, $P(\partial\epsilon/\partial x)$, is Gaussian with unit variance for both unitary and orthogonal ensembles. This prediction has been confirmed by simulation (Fig. 3) over a wide range of velocities. The variation of the conductance as a function of disorder is shown in the inset. It is consistent with the approximate $1/W^2$ dependence of G predicted by the Born approximation.[9]

In conclusion, we have argued that the dispersion of the spectra of quantum chaotic systems in response to an arbitrary external perturbation depends on two parameters: the mean level spacing, Δ, and the generalized conductance, $C(0)$. We have proposed and verified a rescaling in which the dependence on these parameters can be eliminated. We have suggested that the universality applies to all classes of chaotic systems with the same generality as the Wigner–Dyson distribution.

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