Black-hole solution in 2D gravity with torsion

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The 2D model of gravity with zweibeins e^a and with the Lorentz connection one-form ω^a_b as independent gravitational variables is considered. It is shown that the classical equations of motion are exactly integrated in the coordinate system determined by the components of 2D torsion. For some choice of integration constant the solution is of the charged black hole type. The conserved charge and ADM mass of the black hole are calculated.

Considerable attention has recently been given to the study of two-dimensional dilaton gravity. This study has been mainly inspired by string theory, and also by the fact that it gives the simplest model for the dynamical description of a two-dimensional gravity.[1–5] The gravitational variables are the dilaton and metric fields (ϕ, $g_{\mu\nu}$). In empty (without matter) space the classical equations of motion are exactly integrated[1–3] and the solution describes the two-dimensional black hole. On the quantum level, it was shown[4] that this model is renormalizable. One can consider the 2D dilaton gravity as a "toy model" for the study of old problems of black hole formation and evaporation.[5]

On the other hand, numerous recent attempts to formulate the theory of gravity in the framework of a consistent gauge approach resulted in constructing the gauge gravity models for the de Sitter and Poincaré groups (see, e.g., the review article in Ref. 6). The independent variables are now vielbeins $e^a=e^a_\mu dx^\mu$ and Lorentz connection one-form $\omega^a_b=\omega^a_{b,\mu}dx^\mu$. The application of these methods in two dimensions was justified by attempts to give an alternative description of two-dimensional dynamic gravity in terms of the variables (e^a,ω^a_b). It was argued that investigation of a simple two-dimensional model leads to a better understanding of four-dimensional gravity and its quantization.[7] It was shown[7] that the Lagrangian $L=\gamma R^2+\beta T^2+\lambda$ is the most general Lagrangian which is quadratic in curvature R and torsion T, and which contains a cosmological constant λ. The classical equations of motion were analyzed in the conformal gauge[7] and in the light-cone gauge[8] and their exact integrability was demonstrated.

In this note we will consider the model for two-dimensional de Sitter gravity. The constants γ, β, and λ are fixed in this case with only one free parameter α^2 and the action is of the Yang–Mills type.[6]

1. In two dimensions the gauge gravity is described in terms of zweibeins $e^a=e^a_\mu dz^\mu$, $a=0.1$ (the 2D metric on the surface M^2 has the form $g_{\mu\nu}=e^a_\mu e^b_\nu\eta_{ab}$) and Lorentz connection one-form $\omega^a_b=\omega\epsilon^a_b$, $\omega=\omega_\mu dz^\mu$ ($\epsilon_{ab}=-\epsilon_{ba}$, $\epsilon_{01}=1$). The de Sitter curvature two-form $\mathscr{R}^6$ in two dimensions takes the form

$$\mathscr{R}=\begin{pmatrix} R\epsilon^a_b+\alpha^2 e^a\wedge e_b & \alpha T^a \\ \alpha T_b & 0 \end{pmatrix},$$

where α is the coupling constant, and the curvature and torsion two-forms are $R=d\omega$, $T^a=de^a+\epsilon^a_b\omega\wedge e^b$.

The dynamics of gravitational variables (e^a, ω) is determined by the action of the Yang–Mills type:[6]

$$S=\int_{M^2}\frac{1}{4}T\tau{*}\mathscr{R}\wedge\mathscr{R}=\int_{M^2}\frac{\alpha^2}{2}{*}T^a\wedge T^a+\frac{1}{2}{*}R\wedge R-\frac{\alpha^4}{4}\epsilon_{ab}e^a e^b+\alpha^2 R, \qquad (1)$$

where $*$ is the Hodge dualization. The last term in (1) is the boundary term and it does not affect the equations of motion.

Let us consider the variables $\rho={*}R$ and $q^a={*}T^a$. Variation of action (1) with respect to zweibeins e^a and Lorentz connection ω leads to the following equations of motion:

$$d\rho=-\alpha^2 q^a\epsilon_{ab}e^b \qquad (2)$$

$$\nabla q^a=-\frac{1}{2\alpha^2}\left[\rho^2+\alpha^2 q^2-\alpha^4\right]\epsilon^a_b e^b, \qquad (3)$$

where $\nabla q^a\equiv dq^a+\omega\epsilon^a_b q^b$.

2. One particular solution of (2) and (3) is evident. Assuming $q^2=$const, we obtain the following expression from (2) and (3), provided e^a are linearly independent everywhere on M^2: $\rho^2=\alpha^4$, $q^a=0$ at all points of the two-dimensional manifold. In other words, the torsion is zero and M^2 is the de Sitter space.

Let now q^2 be nonconstant, and hence nonzero identically everywhere in M^2. Then from Eqs. (2) and (3) we have the following equation which connects q^2 and ρ:

$$\frac{dq^2}{d\rho}=\frac{1}{\alpha^4}\Phi, \qquad (4)$$

where $\Phi(\rho,q^2)=\rho^2+\alpha^2q^2-\alpha^4$.

The solution of this equation has the form

$$q^2(\rho)=-\frac{1}{\alpha^2}(\rho+\alpha^2)^2+\epsilon e^{\rho/\alpha^2}, \qquad (5)$$

where ϵ is the integrating constant. We will see that it is proportional to the ADM mass. Notice that due to the pseudo-Euclidean signature, q^2 can take both positive and negative values.

We see that for a large negative ρ, regardless of the value of the integration constant ϵ, the function $q^2(\rho)$ has the asymptotics $q^2\sim-(1/\alpha^2)(\rho+\alpha^2)^2$. The form of this function of positive ρ depends on the constant ϵ.

A. $\epsilon>0$.

In this case for a large positive ρ the function q^2 is positive and is approximately equal to $q^2 \sim \epsilon e^{(\rho/\alpha^2)}$.

The critical points of the function $q^2(\rho)$ (5) (where $dq^2/d\rho=0$) are the solutions of the equation

$$\rho_c = -\alpha^2 + \frac{\epsilon}{2} e^{\rho_c/\alpha^2}. \tag{6}$$

It can be shown that there are no such points for $\epsilon > 2\alpha^2$; for $\epsilon = 2\alpha^2$ we obtain one critical point $\rho_c = 0$; for $0 < \epsilon < 2\alpha^2$ the function has two critical points: the first one is positive ($\rho_{c1} > 0$) and the second is negative ($\rho_{c2} < 0$).

In general, $q^2(\rho)$ at the critical point is equal to the following value:

$$q_c^2 = \frac{\epsilon}{2} e^{\rho_c/\alpha^2} \left(1 - \frac{\rho_c}{\alpha^2}\right). \tag{7}$$

We see that q_c^2 is positive if $\rho_c < 0$ (since $\epsilon > 0$). The sign of q^2 at the positive critical point ρ_{c1} depends on the value of the constant ϵ. If ϵ is slightly smaller than $2\alpha^2$, then q_{c1}^2 is still positive. The point ρ_{c1} is a minimum which decreases with decreasing constant ϵ and reaches zero value $q_{c1}^2 = 0$ if, as follows from (7), $\rho_{c1} = \alpha^2$. We can see from (6) that it corresponds to[1)] $\epsilon = 4\alpha^2/e$. Thus we come to the following conclusion about the behavior of the function $q^2(\rho)$.

For $\epsilon > 4\alpha^2/e$ the function $q^2(\rho)$ has only one zero at a negative $\rho < -\alpha^2$. If $\epsilon = 4\alpha^2/e$, then there are two such zeros: at $\rho < -\alpha^2$ and $\rho = \alpha^2 > 0$. For $0 < \epsilon < 4\alpha^2/e$ the function $q^2(\rho)$ vanishes at three points: one for $\rho < -\alpha^2$ and two for $\rho > -\alpha^2$ (one of which satisfies $\rho > \alpha^2$).

B. $\epsilon = 0$.

In this case the function (5) reduces to $q^2 = -(1/\alpha^2)(\rho + \alpha^2)^2$, which is negative everywhere except at the point $\rho = -\alpha^2$, where it vanishes.

C. $\epsilon < 0$.

As we can see from (5), the function $q^2(\rho)$ has no zeros in this case and it is negative for any ρ. Evidently there is only one critical point (the maximum) ρ_c which lies in the interval $-\alpha^2 - (|\epsilon|/e) < \rho_c < -\alpha^2$.

3. Thus, Eqs. (2) and (3) determine $q(=\sqrt{q^2})$ as a function of ρ. Further analysis of (2) easily shows that $\xi(q) = 0$, where we denoted one-form $\xi = q_c e^c$. We will use it and introduce a new coordinate system which is (pseudo)polar with q playing the role of a 'radial' coordinate, while the 'angular' coordinate ϕ is then clearly such that its differential is proportional to ξ. Assuming (for definiteness) that $q^2 = (q^0)^2 - (q^1)^2 > 0$, we can write the torsion components in the form $q^0 = q \cosh \phi$, $q^1 = q \sinh \phi$.

Let us assume that q, ϕ are the new local coordinates on M^2. The differentials $\{dq, d\phi\}$ form the basis in the space of one-forms. Since q is a function of ρ, we can use an equivalent basis $\{d\rho, d\phi\}$. From the construction of q, ϕ (see above) and from (2) and (3) we obtain

$$q^a \epsilon_{ab} e^b = -\frac{d\rho}{\alpha^2},$$
$$q_a e^a = \xi = B d\phi, \tag{8}$$

where B is some function of the variables ρ and ϕ.

Solving Eqs. (2) and (3), we finally find

$$B = q^2 B_0 \exp\left(-\frac{\rho}{\alpha^2}\right),$$

where B_0 is an arbitrary function of ϕ. Consequently, the metric has the form

$$d^2s = (e^0)^2 - (e^1)^2 = q^2(\rho)\exp\left(-\frac{2\rho}{\alpha^2}\right)(d\phi)^2 - \frac{1}{\alpha^4 q^2(\rho)}(d\rho)^2, \tag{9}$$

where $q^2(\rho)$ is a known function (5). Without loss of generality we have redefined the "angular" variable $B_0(\phi)d\phi \to d\phi$.

4. We recall that when the action is varied in order to obtain an equation of motion, one usually drops out the surface term which arises when integrating by parts. The correct way of doing this is to impose appropriate boundary conditions. Assuming the variations $\delta\omega$ and δe^a are arbitrary at spatial infinity (which in the polar coordinate system corresponds to the infinite value of the usual radial coordinate), we determine from the action (1) the boundary conditions

$$\rho|_\infty = -\alpha^2; \quad q^a|_\infty = 0. \tag{10}$$

The constraint that torsion at spatial infinity is zero is too strong. It leads to the constraint $\epsilon = 0$ in (5), so most of the solutions are omitted.

Let us add to the action (1) the following term:

$$S_b = -a\alpha^2 \int_{\partial M^2} \gamma^{1/2} d\tau, \tag{11}$$

where $\gamma = \det \gamma_{\mu\nu}$, $\gamma_{\mu\nu} = g_{\mu\nu} - \kappa n_\mu n_\nu$ is a metric induced on the boundary ∂M^2 with a normal vector n_μ $(n_\mu n_\nu g^{\mu\nu} = \kappa)$, and $\kappa = 1$ for a space-like boundary and $\kappa = -1$ for a time-like boundary. The variation of the total action $S_{tot} = S + S_b$ (note that it is still a positive in Euclidean signature for $a > 0$) will then lead to the modified boundary conditions at spatial infinity, which for metric (9) take the form

$$\rho|_\infty = -\alpha^2; \quad q|_\infty = a. \tag{12}$$

This means that the integration constant in (5) $\epsilon = a^2 e$. Thus the solution describes the two-dimensional, asymptotically de Sitter space with two kinds of possible singularities, where $\rho = -\infty$ and $\rho = +\infty$.

5. The most interesting solution is of the type A, where ρ lies in the interval $-\alpha^2 \leqslant \rho < +\infty$. We see that metric (9) describes the two-dimensional, asymptotically de Sitter space-time with a singularity ($\rho = +\infty$) and horizons at the points where the function $q^2(\rho)$ has zeros.

As was described above, for $\epsilon > 4\alpha^2/e$ such points are absent and we have a naked singularity. For $0<\epsilon<4\alpha^2/e$ we obtain two horizons which coincide when $\epsilon=4\alpha^2/e$. Thus the metric (9) for $0<\epsilon\leqslant 4\alpha^2/e$ resembles the charged two-dimensional black-hole solution[3] [although (9) is not exactly the metric considered in Ref. 3 for 2D dilaton gravity coupled with the Maxwell field]. The case $\epsilon=4\alpha^2/e$ corresponds to the extremal black hole.

In support of this analogy we note that Eq. (2) is similar to Maxwell's equation $df=*j$, where $f=*(dA)$ is the strength of the Abelian gauge field A, and j is the charged matter current one-form. The second gravitational equation in (3) will then be similar to the equation of motion for charged matter. This is not surprising because the local Lorentz symmetry in two dimensions is Abelian and analogous to the $U(1)$ symmetry of Maxwell's theory.

From Eq. (3) we find that the corresponding Lorentz current one-form $*J=-\alpha^2 q^a \epsilon_{ab} e^b$ is conserved, $d{*}J=0$. Integrating $*J$ over any space-like hypersurface Σ, we see that the total charge $Q=\int_{\Sigma} *J$ is equal to the curvature ρ at infinity:[2)]

$$Q=\rho|_{\infty}. \tag{13}$$

Consequently, for the boundary conditions (12) the total charge $Q=-\alpha^2$.

6. To calculate the ADM mass for the black hole solution (9), we will assume[3] that only the equation for ω (2) is satisfied and consider the zweibein energy-momentum one-form $T^a=T^a_\mu dz^\mu$, which can be determined as follows: $\delta_e S=\int -*T^a\wedge\delta e^a$. For action (1) it takes the form

$$\tilde{T}^a\equiv -*T^a=\alpha^2\nabla q^a+\tfrac{1}{2}[\rho^2+\alpha^2 q^2-\alpha^4]\epsilon^a_b e^b.$$

Multiplying this expression by $q^a\exp[-(\rho/\alpha^2)]$, we find that

$$T=\tilde{T}^a q^a \exp\left(-\frac{\rho}{\alpha^2}\right)=\alpha^2\exp\left(-\frac{\rho}{\alpha^2}\right)\left[\frac{1}{2}\,dq^2-\frac{1}{2\alpha^4}\,(\rho^2+\alpha^2q^2-\alpha^4)d\rho\right] \tag{14}$$

is obviously conserved: $dT=0$. This implies that there exists such a scalar function m that

$$T=dm. \tag{15}$$

Straightforward calculations show that the mass function m at the point ρ can be written in explicit form:

$$m=\frac{\alpha^2}{2}\exp\left(-\frac{\rho}{\alpha^2}\right)\left[q^2+\frac{1}{\alpha^2}\,(\rho+\alpha^2)^2\right]. \tag{16}$$

In the case where the field equations $*T^a=0$ are satisfied, Eq. (17) implies that $m=\text{const}$ and for $q^2(\rho)$ in the form (5) we see that $m=\alpha^2\epsilon/2$.

Hence only the solution of the type A describes the positive mass configuration (solutions such as B and C have correspondingly zero and negative mass).

[1]$e=2.7...$ is the Euler number.
[2]Note that the same formula is valid in $(1+1)$ electrodynamics:[3] $Q\equiv\int_{\Sigma}{*j}=f|_{\infty}$.

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